VILNIUS TECHNICAL UNIVERSITY
INSTITUTE OF MATHEMATICS AND INFORMATICS

MATHEMATICAL MODELLING FOR TECHNOLOGY PROBLEMS

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MATHEMATICAL MODELLING FOR TECHNOLOGY PROBLEMS

Selected articles

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Leidinyje pateikiami straipsniai, kuriuose nagrinėjami įvairių industrinių uždavinių matematiniai modeliai, pagrįsti diferencialinėmis lygtimis. Tai vibrotechnikos, statybinės mechanikos, ekologijos ir kt. uždavinių matematiniai modeliai.

Straipsnių rinkinys skiriamas matematinio modeliavimo specialistams – matematikams ir inžinieriams.

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NUMERICAL EXPERIMENT IN MODELLING OF LIQUID-METAL CONTACTS

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1. Problem formulation. The paper is devoted to problems of mathematical modelling of shape of static and nonstatic liquid-metal free surface contacts. Such contacts are distinguished by the fast connection, high reliability, stability under vibration, long term use, little contact resistivity. A various form of such contacts could be introduced, but we shall restrict our attention to nonreservoired contacts with fixed places of sticking to contact planes, prescribed volume of liquid metal in the contact (see Ragulskis et.al., 1986). The mathematical modelling of processes in these contacts brings us to investigation of situation of equilibrium of free surface of contact (static problem) or evolution of this surface (dynamic problem). These problems shall be formulated as non classical boundary value problems for the system of nonlinear equations with nonlocal (integral) condition, what leads to usage of variational methods. Two important features of such a methodology may be outlined. First, the constructed difference schemes are conservative, i.e., a discrete analogue of the conservation law is satisfied exactly; second, the usage of alternative variation formulation naturally enables us to define a generalised (weak) solution of a boundary value problem (see Raim. Čiegis, Rem. Čiegis, R. Čiupaila, 1992).

The main task of this paper is to show the results of mathematical approach of modelling of two different types of contact state: connected (drop) and disconnected (bridge) contact. For this purpose, theoretical investigation of numerical methods, used for computational investigation are provided. The stand out interests are construction of mathematical models of different cases of liquid-metal contact, construction of a conservative difference schemes, investigation of the existence of its generalised (weak) solutions and the iterative methods for finding the solution and numerical results as well.

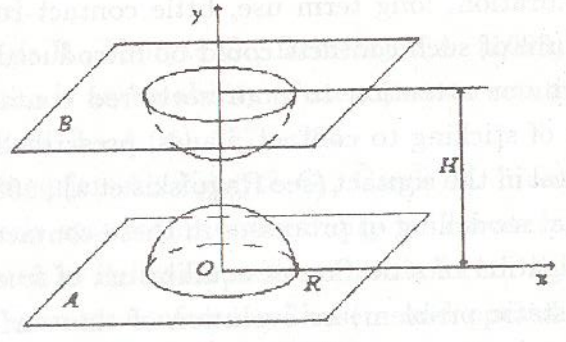

Fig.1.

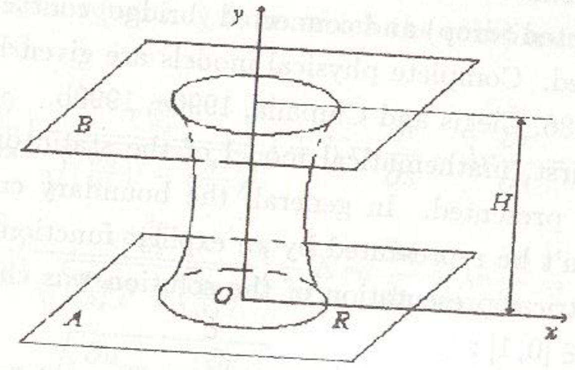

Fig.2.

2. Models and equations. We consider physical models of liquid-metal contact which could be presented in two different conditions: connected contact as a bridge between two contact plates and disconnected contact as a drop lying on the plate or hanging under the plate (see fig. 1,2). The drop of electroconductive nonviscous liquid of prescribed volume V_0 is compressed by two parallel planes and fixed to specially treated disks of radius R. The remaining part of the contact surface is considered free and depends only on a vertical gravity field and surface tension. We restricted our attention to the case, when the electromagnetic forces can be neglected. Using the symmetricity of the problem it is sufficient to investigate only 1D case. The mathematical model follows from the constrained minimisation problem of total energy:

$$\inf_{u \in U} E(u) = E(u^*), \quad E(u) = E_s(u) + E_p(u), \tag{1.1}$$

where $E_p(u), E_s(u)$ are are the gravitational energy and free surface energy, respectively.

Further two different mathematical models of static disconnected (drop) and connected (bridge) contacts shall be considered. Complete physical models are given by Kairytė et.al., 1986, Čiegis and Čiupaila, 1990a, 1990b.

First, mathematical model of the static drop on the plane is presented. In general, the boundary curve of flat drop can't be represented by an explicit function, hence the parametrical presentation of the solution was chosen: $u(s)$, $r(s)$, $s \in [0,1]$:

$$E_s = 2\pi\sigma \int_0^R r(s)(\partial r/\partial s)^2 + (\partial u/\partial s)^2 ds, \tag{1.2}$$

$$E_p = \pi\rho g \int_0^R r(s) \frac{\partial r}{\partial s} u^2 ds.$$

The constancy of volume of the fluid is constraint that must be respected, when the minimum of total energy is determined

$$g(u) = 2\pi \int_0^1 r(s)u(s)\frac{\partial r}{\partial s} ds - V_0 = 0. \tag{1.3}$$

The variational formulation of free surface problem is a general method for solving of such a problem (see Concus and Fin, 1974, Giusti, 1984). Introducing a Lagrange parameter λ and using the necessary conditions of the minimum of the functional of total energy

$$E(u) = E_s(u) + E_p(u) - \lambda(V - V_0) \tag{1.4}$$

we get the differential boundary value problem with an additional nonlocal condition, from which the solutions $u(s), r(s)$ are obtained

$$-\frac{\partial}{\partial s}\left(\frac{r\frac{\partial u}{\partial s}}{\sqrt{(\frac{\partial u}{\partial s})^2 + (\frac{\partial r}{\partial s})^2}}\right) + Kru\frac{\partial r}{\partial s} - \lambda r\frac{\partial r}{\partial s} = 0, \tag{1.5}$$

$$-\frac{\partial}{\partial s}\left(\frac{r\frac{\partial r}{\partial s}}{\sqrt{(\frac{\partial u}{\partial s})^2 + (\frac{\partial r}{\partial s})^2}}\right) - Kru\frac{\partial u}{\partial s} + \lambda r\frac{\partial r}{\partial s} = \sqrt{(\frac{\partial u}{\partial s})^2 + (\frac{\partial r}{\partial s})^2},$$

$$u'(0) = 0, \quad u(1) = 0, r(0) = 0, r(1) = R, \tag{1.6}$$

$$K = \rho g\sigma, \quad \lambda = \lambda'/\sigma.$$

The usual nonparametric case of the problem can be obtained by using the following parametrisation $r = sR$, $\partial r/\partial s = R$ (see Kairytė et al., 1986). The differential boundary value problem with an additional nonlocal condition in this case can be presented as follows

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial u}{\sqrt{1 + (\partial u/\partial r)^2 \partial r}}\right) - Ku - \lambda = 0, \tag{1.7}$$

$$u(R) = 0, \quad u'_r(0) = 0, \quad 2\pi \int_0^R rudr - V_0 = 0. \tag{1.8}$$

This mathematical model is more easy to solve numerically, but as it is shown by Čiegis and Čiupaila, 1990b, such model can be applied for limited set of parameters only.

Mathematical model of static bridge between two parallel planes also follows from the constrained minimisation problem of total energy, where $E_p(u), E_s(u)$ are the gravitational energy and free surface energy, respectively,

$$E_s = 2\pi\sigma \int_0^H u(x)\sqrt{1+u_x^2}dx, \quad E_p = \pi\rho g \int_0^H xu^2(x)dx. \tag{1.9}$$

The constancy of the volume of the fluid is constraint that must be respected, when the minimum of total energy is determined

$$g(u) = \pi \int_0^H u^2(x) - V_0 = 0. \tag{1.10}$$

Differential boundary value problem with an additional nonlocal condition is as follows:

$$-\frac{d}{dx}\left(\frac{u(x)}{\sqrt{1+u_x^2}}\frac{du}{dx}\right) + (Kx+\lambda)u(x) = -\sqrt{1+u_x^2} \tag{1.11}$$

$$u(0) = R,$$

$$u(H) = R \quad g(u) \equiv \pi \int_0^H u^2(x)dx - V_0 = 0. \tag{1.12}$$

Mathematical model of nonstatic contact is more complicated (see Čiegis, Čiupaila, 1991). Under some assumptions the contact can be considered as one-dimensional. The vertical component of the speed $v(y)$ is considered as constant and horizontal component $u(y)$ is considered to be linear function of the radius.

The dynamic equations of evolution of liquid contact can be obtained from the principal of minimal action of functional of action

$$S = \int_0^H (E_k - E_s - E_p)dt, \tag{1.13}$$

where the integrals of kinetic energy, energy of tension and gravitational energy E_k, E_s, E_p can be presented as integrals

$$E_k = \rho \int_0^2 \frac{v^2+u^2}{4} w(\alpha)d\alpha,$$

$$E_p = \rho g \int_0^1 yw(\alpha)d\alpha, \tag{1.14 - 1.15}$$

$$E_s = 2\pi\sigma \int_0^1 \frac{w(\alpha)}{\partial y/\partial\alpha}((\partial x/\partial\alpha)^2 + (\partial y/\partial\alpha)^2)d\alpha, \tag{1.16}$$

Here the parametrisation of domain is introduced by the equations $y = y(\alpha), x = x(y(\alpha)), \alpha \in [0,1]$. The following conditions must be added

$$\pi x^2 \frac{\partial y}{\partial\alpha} = w(\alpha), \ \alpha \in 0,1,$$

$$w(\alpha) \geqslant \varepsilon > 0, \quad \int_0^1 w(\alpha)d\alpha. \tag{1.17 - 1.19}$$

The variational problem of liquid contact evolution is described by the following nonlinear differential equation of the fourth order

$$w(\alpha)\rho\frac{dv}{dt} + \rho\frac{\partial}{\partial\alpha}\left(\frac{1}{4}x\frac{1}{\partial y/\partial\alpha}w(\alpha)\frac{du}{dt}\right) =$$

$$= 2\pi\sigma\Big(\frac{\partial}{\partial\alpha}\Big(\frac{x(\partial y/\partial\alpha)}{\sqrt{(\frac{\partial x}{\partial\alpha})^2 + (\frac{\partial y}{\partial\alpha})^2}}\Big) + \tag{1.20}$$

$$+ \frac{\partial}{\partial\alpha}\Big(\frac{x^3}{2}\frac{\pi}{w(\alpha)}\frac{\partial}{\partial\alpha}\Big(\frac{x(\partial x/\partial\alpha)}{\sqrt{(\frac{\partial x}{\partial\alpha})^2 + (\frac{\partial y}{\partial\alpha})^2}}\Big)\Big) -$$

$$- \frac{\partial}{\partial\alpha}\Big(\frac{x^3}{2}\frac{\pi}{w(\alpha)}\sqrt{(\frac{\partial x}{\partial\alpha})^2 + (\frac{\partial y}{\partial\alpha})^2}\Big)\Big) - \rho g w(\alpha).$$

Condition of indissolubleness (1.18), kinematic conditions (1.21), boundary an initially values are added to complete the formulation of mathematical model

$$dy/dt = v, \quad dx/dt = u, \tag{1.21}$$

$$y(0) = 0, \quad x(0) = R, \quad y(1) = H, \quad x(1) = R, \tag{1.22}$$

$$y(\alpha, t_0) = y_0(\alpha), \quad \nu(\alpha, t_0) = \nu_0(\alpha). \tag{1.23}$$

3. Difference schemes. We propose the following method of construction of conservative difference schemes. In the first step all the energy integrals are approximated by appropriate integration formula. Then the difference scheme follows from the necessary minimum condition for the discrete function

$$E^h(u) = E_s^h(u) + E_p^h(u) - \lambda(v^h - V_0). \tag{2.1}$$

The problem (1.1) is replaced by the discrete constrained minimisation problem. The drop on the plane satisfies finite-difference equations

$$-\Big(\frac{\frac{r_{i+1}+r_i}{2}}{\sqrt{y_s^2 + r_s^2}}y_s\Big)_{\bar{s}} + (Ky_i - \lambda)\frac{r_{i+1}^2 - r_{i-1}^2}{4h} = 0, \tag{2.2}$$

$$-\Big(\frac{\frac{r_{i+1}+r_i}{2}}{\sqrt{y_s^2 + r_s^2}}r_s\Big)_{\bar{s}} - K\frac{y_{i+1}^2 - y_{i-1}^2}{4h} +$$

$$+ \lambda r_i \frac{y_{i+1} - y_{i-1}^2}{2h} + \frac{1}{2}\Big(\sqrt{y_s^2 + r_s^2} + \sqrt{y_{\bar{s}}^2 + r_{\bar{s}}^2}\Big) = 0. \tag{2.3}$$

Discrete boundary and non local conditions are as follows

$$-\frac{\frac{r_1+r_0}{2}}{\sqrt{(\frac{y_1-y_0}{h})^2 - (\frac{r_1-r_0}{h})^2}}\frac{y_1 - y_0}{h}\frac{1}{h} + \frac{1}{2}K\frac{r_1 + r_0}{2}\frac{r_1 - r_0}{2}y_0 -$$

$$- \lambda\frac{r_1^2 - r_0^2}{8} = 0, \tag{2.4}$$

$$r_0 = hR/2, \quad r_N = R, \quad y_N = 0, \tag{2.5}$$

$$2\pi\sum_{i=0}^{N-1} r_{i+1}\frac{y_{i+1} + y_i}{2}\frac{r_{i+1} - r_i}{2}h - V_0 = 0. \tag{2.6}$$

In the case of the simplest parametrisation $r = sR$ this system of equations is reduced to the scheme

$$-\Big(\frac{r}{\sqrt{1 + y_{\bar{r}}^2}}y_{\bar{r}}\Big)_r + Ky_i r_i - \lambda r_i = 0, \tag{2.7}$$

$$y_N = 0, \quad \bar{r}y_{\bar{r}}|_0 = 0, \quad 2\pi\sum_{i=0}^{N-1}\bar{r}\bar{y}h = V_0. \tag{2.8 – 2.9}$$

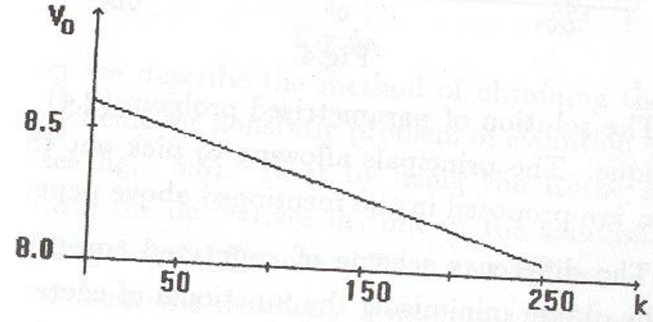

Fig.3.

A possibility to implement such a simple model is restricted by a few relations of parameters V, R, K (see Čiegis, Čiupaila, 1990b). The algorithm allowing to verify the dependence of these parameters to the class of permissible values is given (see fig. 3). The solution of the problem (2.7) - (2.9) exists in the classical sense for the drops with some restricted volume of fluid. With larger volumes of fluid in the drop the generalised (weak) solution exists only (see fig. 4).

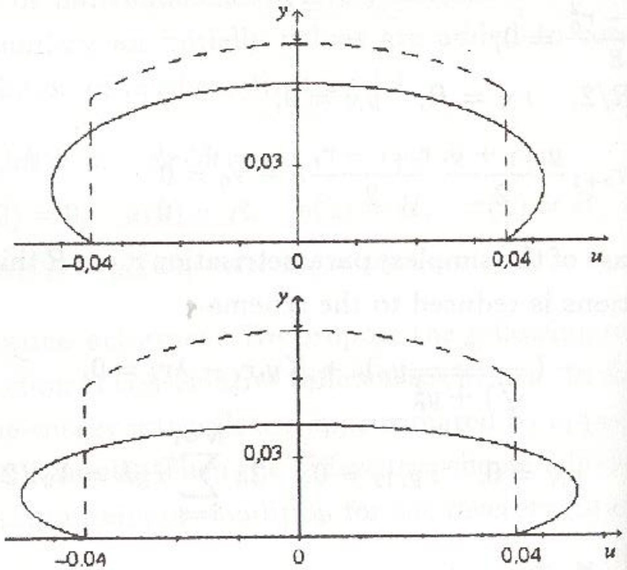

Fig.4.

The solution of parametrised problem (2.4) - (2.6) is not unique. The principals allowing to pick out the unique solution are proposed in the mentioned above paper.

The difference scheme of connected contact (bridge) could be obtain minimising the functional of energy, as well. We get the conservative finite-difference scheme with the additional nonlocal condition

$$-\Big(\frac{\sigma\bar{u}}{\sqrt{1+u_{\bar{y}}^2}}u_{\bar{y}}\Big)_y+\frac{1}{2}(\rho g\bar{y}_i+\lambda)\bar{u}_i+$$
$$+\frac{1}{2}(\rho g\bar{y}_{i-1}+\lambda)\bar{u}_{i-1}=$$
$$=-\frac{\sigma}{2}\Big(\sqrt{1+u_{\bar{y}}^2}+\sqrt{1+u_y^2}\Big), \qquad (2.10)$$
$$u_0=R, \quad u_N=R, \quad \pi\sum_{i=0}^{N-1}\bar{u}^2h-V_0=0. \qquad (2.11)$$

A Taylor expansion reveals that the approximation error of all the difference schemes considered here is of order $O(h^2)$, i.e. it is the same as the approximation accuracy of the energetic integrals by discrete sums.

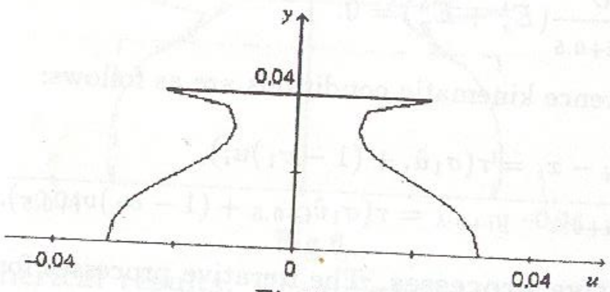

Fig.5a.

Next we describe the method of obtaining the finite-difference scheme for nonstatic problem of evolution of liquid contact (see fig. 5,6). First by using the Rothe method, we substitute the derivatives in time of the solution by the corresponding finite-differences. Then the difference scheme is obtained from the necessary minimum condition for the discrete functional of action

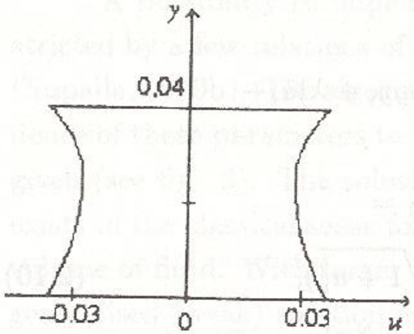

Fig.5b.

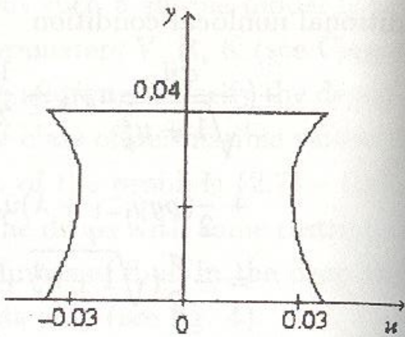

Fig.5c.

$$\frac{\partial S^h_{n+1}}{\partial \hat{y}_{i+0.5}} = 0, \quad i = 1, 2, \cdots, N-1 \qquad (2.12)$$

$$\rho w_i \frac{1}{\tau}((\hat{v}_{i+0.5} - v_{i+0.5}) + 0.5((\hat{u}_{i+1} - u_{i+1})\frac{\partial u_{i+1}}{\partial \hat{y}_{i+0.5}}))+$$

$$+ \frac{\partial}{\partial \hat{y}_{i+0.5}}(\hat{E}^h_s + \hat{E}^h_p) = 0. \qquad (2.13)$$

The difference kinematic conditions are as follows:

$$\hat{x}_i - x_i = \tau(\sigma_1 \hat{u}_i + (1-\sigma_1)u_i),$$

$$\hat{y}_{i+0.5} - y_{i+0.5} = \tau(\sigma_1 \hat{v}_{i+0.5} + (1-\sigma_1)v_{i+0.5}). \qquad (2.14)$$

4. Iterative processes. The iterative processes for solving the nonlinear difference schemes presented in this paper are investigated by Čiegis, Čiupaila, 1990a, 1990b, 1991, 1992. The modified Newton's method was used for the connected and disconnected static contacts and for the evolution processes as well. The two-stage iterative process was proposed to find the free surface of the bridge

$$L_1(y^k)y^{k+1} + \lambda y^{k+1} = -\Psi(y^k), \quad g^h(y^{k+1}) = 0, \qquad (3.1)$$

$$L_1(y)\nu \equiv -(\frac{\sigma y}{\sqrt{1+y^2_{\bar{x}}}} v_{\bar{x}})_x + \rho g x_i \nu_i,$$

$$\Psi(y) = 0.5\sigma(\sqrt{1+y^2_{\bar{x}}} + \sqrt{1+y^2_x}).$$

It is necessary to solve the linear boundary value problem with the nonlocal additional condition in order to find a new iteration y^{k+1}. For this purpose the inner iterative process is used, for example, the bisection method can be used. The convergence of all these iterative processes are shown in the papers mentioned above.

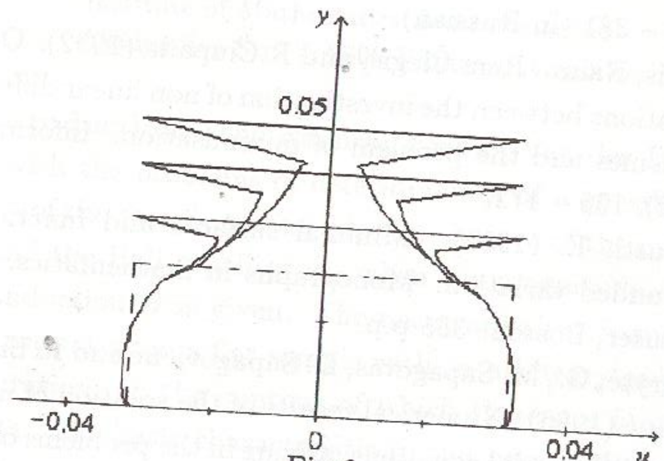

Fig.6.

5. Numerical results. The difference schemes mentioned above were used to obtain the free surface of static drop both on and under the plane, to simulate the statics of disconnection of the bridge into two separate drops. The processes of disconnection in evolution were also investigated.

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