VILNIUS GEDIMINAS TECHNICAL UNIVERSITY

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ON ESTIMATION OF THE HURST INDEX OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

DOCTORAL DISSERTATION

PHYSICAL SCIENCES,
MATHEMATICS (01P)

VGTU
LEIDYKLA
Vilnius TECHNIKA 2011

Doctoral dissertation was prepared at Vilnius Gediminas Technical University in 2007–2011.

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VILNIAUS GEDIMINO TECHNIKOS UNIVERSITETAS

Dmitrij MELICHOV

APIE STOCHASTINIŲ DIFERENCIALINIŲ LYGČIŲ SPRENDINIŲ HURSTO INDEKSO VERTINIMĄ

DAKTARO DISERTACIJA

FIZINIAI MOKSLAI,
MATEMATIKA (01P)

Vilnius LEIDYKLA TECHNIKA 2011

Disertacija rengta 2007–2011 metais Vilniaus Gedimino technikos universitete.

Mokslinis vadovas
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http://leidykla.vgtu.lt
VGTU leidyklos TECHNIKA 1917-M mokslo literatūros knyga

ISBN 978-9955-28-999-9

Abstract

The main topic of this dissertation is the estimation of the Hurst index H of the solutions of stochastic differential equations (SDEs) driven by the fractional Brownian motion (fBm).

Firstly, the limit behavior of the first and second order quadratic variations of the solutions of SDEs driven by the fBm is analyzed. This yields several strongly consistent estimators of the Hurst index H. Secondly, it is proved that in case the solution of the SDE is replaced by its Milstein approximation, the estimators remain strongly consistent. Additionally, the possibilities of applying the increment ratios (IR) statistic based estimator of H originally obtained by J. M. Bardet and D. Surgailis in 2010 to the fractional geometric Brownian motion are examined. Furthermore, this dissertation derives the convergence rate of the modified Gladyshev's estimator of the Hurst index to its real value.

The estimators obtained in the dissertation were compared with several other known estimators of the Hurst index H, namely the naive and ordinary least squares Gladyshev and η-summing oscillation estimators, the variogram estimator and the IR estimator. The models chosen for comparison of these estimators were the fractional Ornstein-Uhlenbeck (O-U) process and the fractional geometric Brownian motion (gBm). The initial inference about the behavior of these estimators was drawn for the O-U process which is Gaussian, while the gBm process was used to check how the estimators behave in a non-Gaussian case. The scope of modelling was 100 sample paths of the length $n = 2^{14} + 1$ for each value of $H \in \{0.55, 0.6, \ldots, 0.95\}$ on the unit interval $t \in [0, 1]$.

The dissertation consists of the introduction, 3 main chapters, the conclusions, the bibliography, the list of author's publications on the topic of dissertation and two appendices.

The results obtained during the doctoral studies were published in 6 papers in reviewed periodic scientific journals and were presented at 5 conferences, of which 2 – international.

Santrauka

Pagrindinė šios disertacijos tema – stochastinių diferencialinių lygčių (SDL), valdomų trupmeninio Brauno judesio (tBj), sprendinių Hursto indekso H vertinimas.

Pirmiausia disertacijoje išnagrinėta SDL, valdomų tBj, sprendinių pirmos ir antros eilės kvadratinių variacijų ribinė elgsena. Iš šių rezultatų seka keli stipriai pagrįsti Hursto indekso H įvertiniai. Įrodyta, kad šie įvertiniai išlieka stipriai pagrįsti, jei tikra sprendinio trajektorija keičiama jos Milšteino aproksimacija. Taip pat išnagrinėtos pokyčių santykio (*increment ratios*) statistikos H įvertinio, gauto J. M. Bardeto ir D. Surgailio 2010 m., taikymo trupmeninio geometrinio Brauno judesio Hursto indekso vertinimui galimybės bei nustatytas modifikuoto Gladyševo H įvertinio konvergavimo į tikrąją parametro reikšmę greitis.

Gauti įvertiniai palyginti su kai kuriais kitais žinomais Hursto indekso H įvertiniais: naiviais bei mažiausių kvadratų Gladyševo ir η-sumavimo osciliacijos įvertiniais, variogramos įvertiniu ir pokyčių santykio statistikos įvertiniu. Įvertinių elgsena buvo palyginta trupmeniniam Ornšteino-Ulenbeko (O-U) procesui bei trupmeniniam geometriniam Brauno judesiui (gBj). Pradinės išvados buvo padarytos O-U procesui, kuris yra Gauso, o gBj procesas buvo naudojamas patikrinti, kaip šie įvertiniai elgiasi, kai procesas yra ne Gauso. Modeliavimo apimtis buvo po 100 trajektorijų kiekvienai Hursto indekso reikšmei $H \in \{0,55,\ 0,6,\ \ldots,\ 0,95\}$ vienetiniame intervale $t \in [0,\ 1]$; kiekvienos trajektorijos ilgis buvo $n = 2^{14} + 1$ taškų.

Disertaciją sudaro įvadas, 3 pagrindiniai skyriai, išvados, literatūros sąrašas, autoriaus publikacijų disertacijos tema sąrašas ir du priedai.

Doktorantūros studijų metu gauti rezultatai buvo paskelbti 6 straipsniuose recenzuojamuose periodiniuose mokslo leidiniuose ir pristatyti 5 konferencijose, iš kurių 2 – tarptautinės.

Notation

Symbols

fBm	– the fractional Brownian motion
SDE	– stochastic differential equation
gBm	– the fractional geometric Brownian motion
O-U	– the Ornstein-Uhlenbeck process
$\mathbb{N}$	– the set of natural numbers
$\mathbb{Z}$	– the set of integer numbers
$\mathbb{R}$	– the set of real numbers
EX	– the expectation of X
MSE(X)	– the mean squared error of X
a.s.	– almost surely
$\mathbf{1}_A$	– the indicator function of the set A
$[x]$	– the integer part of x
$V_n^{(1)}(X,2)$	– the first order quadratic variation of X (regular subdivisions)
$V_n^{(2)}(X,2)$	– the second order quadratic variation of X (regular subdiv.)
$V_{\pi_n}^{(1)}(X,2)$	– the first order quadratic variation of X (irregular subdiv.)
$V_{\pi_n}^{(2)}(X,2)$	– the second order quadratic variation of X (irregular subdiv.)

$v_p(f;[a,b])$ – the p-variation of f on the interval $[a,b]$
$\mathcal{W}_p([a,b])$ – the set of functions with bounded p-variation on $[a,b]$

Contents

Introduction

Formulation of the problem

In fields as diverse as economics and finance, mathematics, physics, chemistry, environmental studies and computer science it is not uncommon to encounter observations made far apart in time or space which are non-trivially correlated. This phenomenon is known as *long memory* or *long-range dependance* and was first studied by the hydrologist Hurst (1951) who tried to derive a suitable model for the flow of the Nile river. The stochastic calculus of stochastic processes possessing the long-range dependance property started with the work of Mandelbrot, van Ness (1968) which introduced the fractional Brownian motion (fBm), the backbone of such processes. Later on B. B. Mandelbrot summarized his results on fractals and scaling in Mandelbrot (1995). The first result in which the fBm appeared as the limit of stationary sums of random variables in the Skorokhod topology was obtained by Taqqu (1975). In the 1990s intensive studies of possibilities of applying the fBm in various teletraffic, finance and climate models started which, in turn, encouraged the stochastic analysis studies of the fBm (e. g., Decreusefond, Üstünel (1995)).

The Hurst index $H \in (0, 1)$ determines the correlation structure of fBm: if $H = 1/2$ it is the standard Brownian motion, if $H < 1/2$ the increments of the process are negatively correlated and if $H > 1/2$ the increments of the process are positively correlated which implies the long-range dependance. The latter property of the fBm encouraged the studies of stochastic models in which the standard Brownian motion is replaced by the fBm, since the long-range dependance is often encountered in the observed data. Therefore it is important to be able to study this dependance and check if it really exists. The problem examined in this thesis is the estimation of the Hurst index H of certain generalizations of the fBm from discrete data.

Topicality of the work

The estimation and modeling of the Hurst index has been a subject of intense studies lately. A whole set of methods and estimators have been proposed for the Gaussian processes of the fractional type. However little is known about the construction of the estimators when the considered process is a solution of a stochastic differential equation driven by the fBm. In the work of Berzin, León (2008) such estimators are given for several specific types of such equations, where the integrands are either constants or linear functions. Naturally it's desirable to obtain estimators for the solutions of the general case of stochastic differential equations driven by the fBm which would be simple to implement and computationally efficient.

Research object

The research objects are the solutions of stochastic differential equations driven by the fractional Brownian motion with the Hurst index $H > 1/2$.

The aim and tasks of the work

The aim of this work is to study the limit behavior of certain statistics based on the observed values of the process and use the obtained results to derive consistent estimators of the Hurst index H as well as to study the properties of these estimators. The tasks of this work are:

1. To study the limit behavior of the quadratic variations of the solutions of SIEs driven by the fBm both in the case of equally and non-equally spaced observations.

2. To derive consistent estimators of the Hurst index H based on quadratic variations.
3. To study the possibility of applying the increment ratios (IR) statistic to estimate the Hurst index H of the solutions of SIEs driven by the fBm.
4. To compare the performance of the obtained estimators to that of other known estimators.

Applied methods

In the theoretical part of the work the p-variation calculus techniques have been applied along with an array of known inequalities. As for the modelling part of the work, the fractional Brownian motion sample paths were generated using the circular matrix embedding method (see, f.e., Coeurjolly (2000)). All calculations were performed using the `R` software package.

Scientific novelty

It was shown that the estimators of the Hurst index H originally obtained by Istas, Lang (1997) and Benassi *et al* (1998) for the fBm retain their properties when the underlying process is a solution of a stochastic differential equation, which is not necessarily Gaussian. Additionally, it was proved that the IR statistic estimator originally obtained by Bardet, Surgailis (2010) can be used to estimate the Hurst index H of the fractional geometric Brownian motion. Furthermore, the convergence rate of the modified Gladyshev Hurst index estimator has been derived.

Practical value of the results

The estimators studied in this work are suitable for a wide spectrum of processes including the fractional Ornstein-Uhlenbeck process and the fractional geometric Brownian motion. For the latter two models, the estimators were additionally studied through simulated data. They are easy to implement, computationally efficient and do not impose any specific requirements on the sample path lengths.

Propositions presented for defence

1. Two strongly consistent estimators of the Hurst index H of the solution of a stochastic differential equation driven by the fractional Brownian motion have been obtained which is a non-covering extension of the results known up to date.
2. It was shown that the IR statistic estimator of the Hurst index H is applicable to the fractional geometric Brownian motion.
3. The convergence speed of the modified Gladyshev estimator has been obtained.
4. Computer modelling suggests that the performance of the obtained estimators is comparable to or better than that of other estimators considered in this non-exhaustive study.

Approval of the results

On the topic of dissertation there were 6 papers published in reviewed scientific journals. The research results were reported at 5 scientific conferences. The list of conference talks is as follows:

1. K. Kubilius, D. Melichov, Estimating the Hurst index of the solution of a stochastic integral equation, *10th international Vilnius conference on probability theory and mathematical statistics*, Vilnius, 2010.
2. K. Kubilius, D. Melichov, On estimation and asymptotics of the Hurst index of solutions of stochastic integral equations, *Applied stochastic models and data analysis*, Vilnius, 2009.
3. K. Kubilius, D. Melichov, Using the IR ir DV statistics to estimate the Hurst index of solutions of stochastic differential equations, *LMD 52nd conference*, Vilnius, 2011.
4. K. Kubilius, D. Melichov, On estimation of the Hurst index of solutions of stochastic integral equations, *LMD 51st conference*, Šiauliai, 2010.
5. K. Kubilius, D. Melichov, On estimation of the Hurst index of solutions of stochastic integral equations, *LMD 50th conference*, Vilnius, 2009.

Structure of the dissertation

The dissertation consists of the introduction, three chapters, the conclusions, references, the list of author's publications on the topic of the dissertation and two appendices. The total scope of the dissertation is 78 pages, 5 tables, 4 figures and 34 items of reference.

The first chapter is the introduction which presents the considered stochastic differential equation, the overview of other authors' works on the topic of dissertation and introduces some common definitions used further on.

The second chapter presents the obtained theoretical results, namely the asymptotics of the quadratic variations of the solutions of stochastic differential equations driven by the fBm and the estimators of the Hurst index H. Additionally, the usage of the IR statistic based estimator to estimate the Hurst index of the solutions of SDEs is considered; it's proved that if the underlying process is the fractional geometric Brownian motion, then the a.s. convergence of the IR statistic holds. Moreover, the convergence rates of the modified Gladyshev estimator are studied.

The third chapter shows the comparison of performance of the obtained estimators of the Hurst index H with that of other known estimators for a Gaussian (fractional Ornstein-Uhlenbeck) and a non-Gaussian (fractional geometric Brownian motion) processes.

1

Definitions and the historical overview

1.1. The fractional Brownian motion

1.1 Definition. *A process* $B_t^H = \{B_t^H; t \geqslant 0\}$ *is a fractional Brownian motion (fBm) with the Hurst index* $H \in (0,1)$ *if it is a continuous centered Gaussian process with the covariance function*

$$\mathbf{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad \forall t, s \geqslant 0.$$

The fractional Brownian motion has the following primary properties:

- *Self-similarity*: for any $a > 0$, the processes $\{a^H B_t^H; t \geqslant 0\}$ and $\{B_{at}^H; t \geqslant 0\}$ have identic probability distributions;
- *Stationary increments*: for any $s > 0$, the processes $\{B_t^H; t \geqslant 0\}$ and $\{B_{t+s}^H - B_s^H; t \geqslant 0\}$ have identic probability distributions;
- *Correlated increments*: for $u < s < t$, the fBm has independent increments only if $H = 1/2$, this corresponds to the standard Brownian

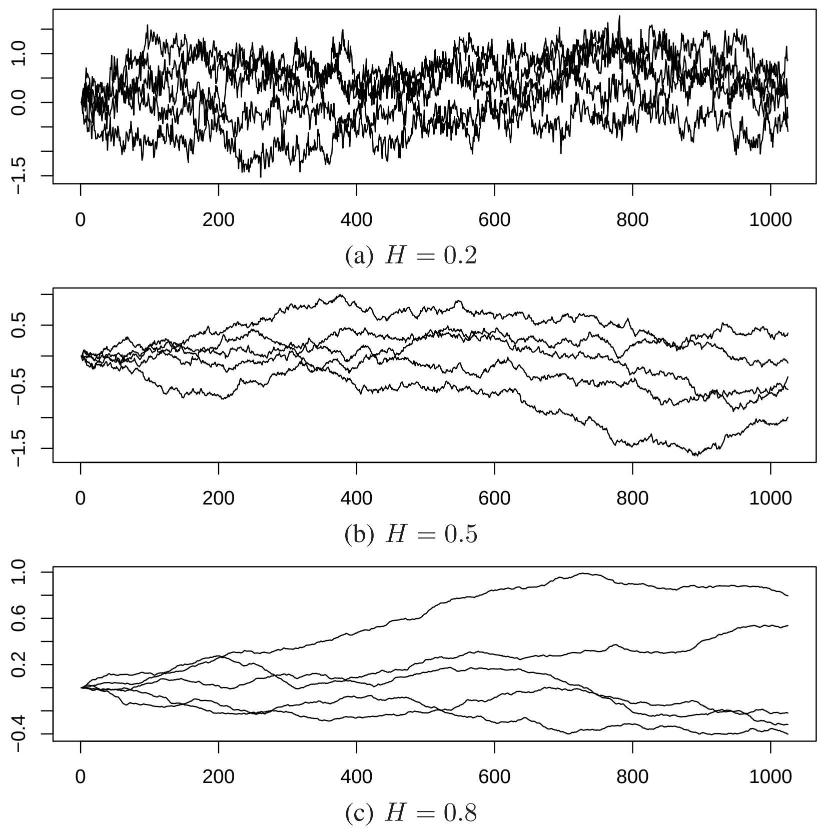

Fig. 1.1. 5 sample paths of the fractional Brownian motion, $n = 1024$.

motion; if $H \neq 1/2$, then the increments of B^H are correlated:

$$corr\left(B_t^H - B_s^H, B_s^H - B_u^H\right) \begin{cases} > 0 & \text{if } H > 1/2; \\ = 0 & \text{if } H = 1/2; \\ < 0 & \text{if } H < 1/2. \end{cases}$$

- *Hölder continuity*: for $\alpha < H$, almost all sample paths of fBm are Hölder continuous of order α, that is,

$$\sup_{t \neq s} \frac{|B_t^H - B_s^H|}{|t - s|^\alpha} < \infty.$$

- *Bounded p-variation*: almost all sample paths of fBm with the Hurst index $1/2 < H < 1$ have bounded p-variation for $p > 1/H$, that is,

$$v_p\left(B^H;[a,b]\right) = \sup_{\varkappa} \sum_{k=1}^{n} \left|B^H(t_k) - B^H(t_{k-1})\right|^p < \infty,$$

  $\varkappa = \{t_i : i = 0, \ldots, n\}$ being all finite partitions of the interval $[a, b]$ such that $a = t_0 < t_i < \ldots < t_n = b$.

- *Fractal dimension*: the graph of a sample path of fBm with the Hurst index H has the fractal dimension equal to $2 - H$.

1.2. The main equation

Consider a SDE driven by the fBm with the Hurst index $1/2 < H < 1$

$$X_t = \xi + \int_0^t f(X_s)\, ds + \int_0^t g(X_s)\, dB_s^H, \tag{1.1}$$

$t \in [0, T]$, $T > 0$, $\xi \in \mathbb{R}$. It is known that almost all sample paths of B^H, $1/2 < H < 1$, have bounded p-variation for $p > 1/H$. Thus the integrals on the right side of (1.1) will exist pathwise as the Riemann-Stieltjes integrals. For $\frac{1}{H} - 1 < \alpha \leqslant 1$, $\mathcal{C}^{1+\alpha}(\mathbb{R})$ denotes the set of all $\mathcal{C}^1$-functions g: $\mathbb{R} \to \mathbb{R}$ such that

$$\sup_x |g'(x)| + \sup_{x \neq y} \frac{|g'(x) - g'(y)|}{|x - y|^\alpha} < \infty.$$

Let f be a Lipschitz-continuos function and let $g \in \mathcal{C}^{1+\alpha}(\mathbb{R})$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. For $1 \leqslant p < 1 + \alpha$ there exists a unique solution of the equation (1.1) with almost all sample paths in the class of all continuous functions defined on $[0, T]$ with bounded p-variation (see Lyons (1994), Dudley (1999), Kubilius (2000), Nualart, Răşcanu (2002)).

For the estimation of the Hurst parameter H first of all we consider the limits of the first and second order quadratic variations of a pathwise solution X of (1.1). Results of such type for Gaussian processes were considered in Bégyn (2005)-Bégyn (2006) (see also references in Bégyn (2006)).

1.3. The estimators of the Hurst index

In 1961, E. Gladyshev derived a limit theorem for a statistic based on the first order quadratic variations of fBm. This yielded an estimator of H which was strongly consistent but not asymptotically normal.

In 1997, another estimator was introduced by J. Istas and G. Lang. This estimator was designed for centered Gaussian processes with stationary increments and again employed the first order quadratic variations. The obtained estimator was asymptotically normal for $H \in (1/2,\, 3/4)$.

In 1998, A. Benassi et al considered the second order quadratic variations of a class of Gaussian processes, having locally the same fractal properties as the fractional Brownian motion and obtained an estimator of the Hurst index H which was asymptotically normal for all H.

In 2001, J. F. Coeurjolly developed a class of consistent estimators of H based on the asymptotic behavior of the k-th absolute moment of discrete variations of its sample paths over a discrete partition of the interval $[0,\, 1]$. Explicit convergence rates for these types of estimators, valid through the whole range $0 < H < 1$ of the self-similarity parameter, were derived, and the asymptotic normality of the obtained estimators was established.

In 2005, A. Bégyn considered the second order quadratic variations along general subdivisions for processes with Gaussian increments. A more complete survey on asymptotic behavior of quadratic variations for Gaussian processes can be found in the thesis of A. Bégyn (2006).

In 2006, C. Berzin and J. R. León proposed estimators of H and the diffusion function g for several specific cases of (1.1), namely for the combinations of $f(X_s) = \mu$ or $f(X_s) = \mu X_s$ and $g(X_s) = \sigma$ or $g(X_s) = \sigma X_s$. Additionally they assumed that the process X_t was smoothed by convolution defined as $X_\varepsilon(t) = \varphi_\varepsilon * X_t$ where ε, which tends to zero, is the smoothing parameter and $\varphi_\varepsilon(\cdot)$ is the convolution kernel defined as $\varphi_\varepsilon(\cdot) = \frac{1}{\varepsilon}\varphi(\frac{\cdot}{\varepsilon})$. Here $\varphi(\cdot)$ is a $\mathcal{C}^2$ positive kernel with L^1 norm equal to one. The estimators of H and g use functionals of the type $\int_0^1 h(X_\varepsilon(t))|\ddot{X}_\varepsilon(t)|^k\, dt$ where $h(x) = 1/|x|^k$ in the case of $g(X_s) = \sigma X_s$ and $h(x) = 1$ in the case of $g(X_s) = \sigma$.

In 2011, R. Bertrand, M. Fhima and A. Guillin introduced a method for change point analysis on the Hurst index for a piecewise fractional Brownian motion, a generalization of the regular fractional Brownian motion. Their procedure is the combination, on the one hand, of the filtered derivative with p-value (FDpV) method for detection of change of the mean, variance or regression parameter, and, on the other hand, of a variation of the increment ratios (IR) statistic estimator introduced in J. M. Bardet and D. Surgailis (2010).

1.4. Quadratic variations

1.4.1. Regular subdivisions

1.2 Definition. *For a real-valued process $X = \{X_t;\, t \in [0,\, T]\}$, we define the first and second order quadratic variations along regular subdivisions as*

$$V_n^{(1)}(X,2) = \sum_{k=1}^{n} (\Delta X_k^n)^2\,, \qquad V_n^{(2)}(X,2) = \sum_{k=1}^{n-1} \left(\Delta^{(2)} X_k^n\right)^2,$$

where

$$\Delta X_k^n = X(t_k^n) - X(t_{k-1}^n)\,, \quad \Delta^{(2)} X_k^n = X(t_{k+1}^n) - 2X(t_k^n) + X(t_{k-1}^n)$$

and $t_k^n = kT/n$.

1.4.2. Irregular subdivisions

Let $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $T > 0$, be a sequence of subdivisions of the interval $[0,\, T]$ and (N_n) is an increasing sequence of natural numbers. Such sequence of subdivisions is called irregular. Define

$$m_n = \max_{1\leqslant k\leqslant N_n} \Delta t_k^n, \qquad p_n = \min_{1\leqslant k\leqslant N_n} \Delta t_k^n, \qquad \Delta t_k^n = t_k^n - t_{k-1}^n.$$

Usually the observed values of the process are only available at discrete regular time intervals. However, it may happen that part of the observations are lost, resulting in observations at irregular time intervals.

1.3 Definition. *The first and second order quadratic variations of X along the subdivisions $(\pi_n)_{n\in\mathbb{N}}$ with normalization $1/2 < H < 1$ is defined by*

$$V_{\pi_n}^{(1)}(X,2) = \sum_{k=1}^{N_n} \frac{(\Delta X_k^n)^2}{(\Delta t_k^n)^{2H-1}}\,, \qquad \Delta X_k^n = X(t_k^n) - X(t_{k-1}^n),$$

and

$$V_{\pi_n}^{(2)}(X,2) = 2 \sum_{k=1}^{N_n-1} \frac{\Delta t_{k+1}^n (\Delta_{ir}^{(2)} X_k^n)^2}{(\Delta t_k^n)^{1/2+H} (\Delta t_{k+1}^n)^{1/2+H} (\Delta t_k^n + \Delta t_{k+1}^n)}\,,$$

where

$$\Delta^{(2)}_{ir} X^n_k = \Delta t^n_k X(t^n_{k+1}) + \Delta t^n_{k+1} X(t^n_{k-1}) - (\Delta t^n_k + \Delta t^n_{k+1}) X(t^n_k).$$

1.5. Auxiliary results

Let $\mathcal{W}_p([a,b])$ denote the set of functions which have bounded p-variation on the interval $[a,\, b]$:

$$\mathcal{W}_p([a,b]) := \{f : [a,\, b] \to \mathbb{R} : v_p\,(f;[a,b]) < \infty\}\,,$$

where

$$v_p\,(f;[a,b]) = \sup_{\varkappa} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p\,,$$

$\varkappa = \{x_i : i = 0,\, \ldots,\, n\}$ being all finite partitions of $[a,\, b]$ such that $a = x_0 < x_i < \ldots < x_n = b$. Let $V_p(f) := V_p(f;[a,b]) = v_p^{1/p}\,(f;[a,b])$. $V_p(f)$ is a non-increasing function of p, that is, if $0 < q < p$ then $V_p(f) \leqslant V_q(f)$. Let $a < c < b$ and let $f \in \mathcal{W}_p([a,b])$ with $0 < p < \infty$. Then

$$v_p(f;[a,c]) + v_p(f;[c,b]) \leqslant v_p(f;[a,b]), \tag{1.2}$$

$$V_p(f;[a,b]) \leqslant V_p(f;[a,c]) + V_p(f;[c,b]). \tag{1.3}$$

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$. The Love-Young inequality states that

$$\left| \int_a^b f\, dh - f(y)\big[h(b) - h(a)\big] \right| \leqslant C_{p,q} V_q(f;[a,b]) V_p(h;[a,b]) \tag{1.4}$$

and

$$V_p\biggl(\int_a^{\cdot} f\, dh;[a,b]\biggr) \leqslant C_{p,q} V_{q,\infty}(f;[a,b]) V_p(h;[a,b]), \tag{1.5}$$

where $V_{q,\infty}(f;[a,b]) = V_q(f;[a,b]) + \sup_{a\leqslant x\leqslant b} |f(x)|$, $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$. Let $f \in \mathcal{W}_q([a,b])$ and $g \in \mathcal{W}_p([a,b])$, $0 < p < \infty$. Then $fg \in \mathcal{W}_p([a,b])$ and

$$V_{p,\infty}(fg;[a,b]) \leqslant C_p V_{p,\infty}(f;[a,b]) V_{p,\infty}(g;[a,b]). \tag{1.6}$$

Let $f \in \mathcal{W}_q([a,b])$ and $g \in \mathcal{W}_p([a,b])$. For any partition $\varkappa$ and for $p^{-1}+q^{-1} \geqslant 1$ the Young's version of Hölder's inequality and the inequality (1.2) yield

$$\sum_i V_q(f;[x_{i-1},x_i])V_p(g;[x_{i-1},x_i]) \leqslant V_q(f;[a,b])V_p(g;[a,b]), \tag{1.7}$$

since

$$\begin{aligned}
&\sum_i V_q(f;[x_{i-1},x_i])V_p(g;[x_{i-1},x_i]) \\
&\quad \leqslant \left(\sum_i V_q^q(f;[x_{i-1},x_i])\right)^{\frac{1}{q}} \cdot \left(\sum_i V_p^p(g;[x_{i-1},x_i])\right)^{\frac{1}{p}} \\
&\quad = \left(\sum_i v_q(f;[x_{i-1},x_i])\right)^{\frac{1}{q}} \cdot \left(\sum_i v_p(g;[x_{i-1},x_i])\right)^{\frac{1}{p}} \\
&\quad \leqslant (v_q(f;[a,b]))^{\frac{1}{q}} \cdot (v_p(g;[a,b]))^{\frac{1}{p}} = V_q(f;[a,b])V_p(g;[a,b]).
\end{aligned}$$

Since almost all sample paths of the B^H, $1/2 \leqslant H < 1$, are locally Hölder continuous, it follows that

$$V_p(B^H;[s,t]) \leqslant L_T^{H,1/p}\,(t-s)^{1/p}, \tag{1.8}$$

where $s < t \leqslant T$, $p > 1/H$,

$$L_T^{H,\gamma} = \sup_{\substack{s \neq t \\ s,t \leqslant T}} \frac{|B_t^H - B_s^H|}{|t-s|^\gamma}, \quad 0 < \gamma < H, \qquad \mathbf{E}\big(L_T^{H,\gamma}\big)^k < \infty, \quad \forall k \geqslant 1.$$

1.6. Conclusions of the first chapter

The estimation of the Hurst index H has been thoroughly studied for various types of Gaussian processes. The goal of this dissertation is to address such estimation when the underlying process is the solution of the stochastic differential equation (1.1) which is not necessarily Gaussian.

2

Quadratic variations and the increment ratios statistic

2.1. Regular subdivisions

2.1 Theorem. *Let f be a Lipschitz-continuos function and let $g \in \mathcal{C}^{1+\alpha}$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. Assume that the subdivision of the interval $[0,\, T]$ is regular. Then*

$$\lim_{n\to\infty} n^{2H-1} V_n^{(1)}(X,2) = \int_0^T g^2(X_t)\, dt$$

where X is the solution of the equation (1.1).

Define

$$\widehat{H}^n_{dv1} := \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}^{(1)}(X,2)}{V_n^{(1)}(X,2)}.$$

Here and further $V_{2n}^{(\cdot)}(X,2)$ corresponds to the quadratic variation of the whole sample path while $V_n^{(\cdot)}(X,2)$ is the variation of the subset $\{X_k : k = 2j, 0 \leqslant j \leqslant [n/2]\}$, $[x]$ denotes the integer part of x.

2.2 Theorem. *Assume that conditions of Theorem* 2.1 *are satisfied. Then*

$$\widehat{H}^n_{dv1} \longrightarrow H \quad \textit{a.s.} \qquad \textit{as } n \to \infty.$$

2.3 Theorem. *Let f be a Lipschitz-continuos function and let $g \in \mathcal{C}^{1+\alpha}$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. Assume that the subdivision of the interval $[0, T]$ is regular. Then*

$$\lim_{n\to\infty} n^{2H-1} V_n^{(2)}(X,2) = (4 - 2^{2H}) \int_0^T g^2(X_t)\, dt$$

where X is the solution of (1.1).

Define

$$\widehat{H}^n_{dv2} := \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}^{(2)}(X,2)}{V_n^{(2)}(X,2)}.$$

2.4 Theorem. *Assume that conditions of Theorem* 2.3 *are satisfied. Then*

$$\widehat{H}^n_{dv2} \longrightarrow H \quad \textit{a.s.} \qquad \textit{as } n \to \infty.$$

The proof of Theorem 2.1 does not differ significantly from that of Theorem 2.6 and shall be omitted. The proof of Theorem 2.2 follows immediately from the proof of Theorem 2.4 and the result of Theorem 2.1.

Proof of Theorem 2.3. Under the conditions of the theorem the solution of the equation (1.1) exists for $1 \leqslant p < 1+\alpha$ and $V_p(X; [0,T]) < \infty$ for every $p > 1/H$. For simplicity the index n for t in the sequel will be omitted. So

$$\begin{aligned}
V_n^2(X,2) =& \sum_{k=1}^{n-1} \left(\int_{t_k}^{t_{k+1}} f(X_s)\, ds - \int_{t_{k-1}}^{t_k} f(X_s)\, ds \right)^2 \\
&+ \sum_{k=1}^{n-1} \left(\int_{t_k}^{t_{k+1}} g(X_s)\, dB_s^H - \int_{t_{k-1}}^{t_k} g(X_s)\, dB_s^H \right)^2 \\
&+ 2 \sum_{k=1}^{n-1} \left(\int_{t_k}^{t_{k+1}} g(X_s)\, dB_s^H - \int_{t_{k-1}}^{t_k} g(X_s)\, dB_s^H \right) \\
&\quad \times \left(\int_{t_k}^{t_{k+1}} f(X_s)\, ds - \int_{t_{k-1}}^{t_k} f(X_s)\, ds \right) \\
=:& S_1 + S_2 + S_{12}.
\end{aligned}$$

Denote $X_k := X(t_k^n)$. Let's evaluate the behavior of

$$\begin{aligned}
&V_n^{(2)}(X,2) - \sum_{k=1}^{n-1} g^2(X_k)\left(\Delta^{(2)} B_k^H\right)^2 \\
&\quad = S_1 + \left(S_2 - \sum_{k=1}^{n-1} g^2(X_k)\left(\Delta^{(2)} B_k^H\right)^2\right) + S_{12} \\
&\quad =: S_1 + \widetilde{S_2} + S_{12}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} g(X_s)\, dB_s^H - \int_{t_{k-1}}^{t_k} g(X_s)\, dB_s^H \\
&= \left(\int_{t_k}^{t_{k+1}} g(X_s)\, dB_s^H - g(X_k)\Delta B_{k+1}^H\right) \\
&\quad - \left(\int_{t_{k-1}}^{t_k} g(X_s)\, dB_s^H - g(X_k)\Delta B_k^H\right) + g(X_k)\Delta^{(2)} B_k^H.
\end{aligned}$$

Thus

$$\begin{aligned}
|\widetilde{S_2}| \leqslant &\sum_{k=1}^{n-1}\left(\int_{t_k}^{t_{k+1}} g(X_s)\, dB_s^H - g(X_k)\Delta B_{k+1}^H\right)^2 \\
&+ \sum_{k=1}^{n-1}\left(\int_{t_{k-1}}^{t_k} g(X_s)\, dB_s^H - g(X_k)\Delta B_k^H\right)^2 \\
&+ 2\sum_{k=1}^{n-1} |g(X_k)\Delta^{(2)} B_k^H| \cdot \left|\int_{t_k}^{t_{k+1}} g(X_s)\, dB_s^H - g(X_k)\Delta B_{k+1}^H\right| \\
&+ 2\sum_{k=1}^{n-1} |g(X_k)\Delta^{(2)} B_k^H| \cdot \left|\int_{t_{k-1}}^{t_k} g(X_s)\, dB_s^H - g(X_k)\Delta B_k^H\right|.
\end{aligned}$$

Further, from the Love-Young inequality (1.4) it follows that

$$\begin{aligned}
|\widetilde{S_2}| \leqslant &\, 2C_{p,p}^2 |g'|_\infty^2 \sum_{k=0}^{n-1} V_p^2(X;[t_k,t_{k+1}]) V_p^2(B^H;[t_k,t_{k+1}]) \\
&+ 2C_{p,p}|g'|_\infty \sum_{k=1}^{n-1} |g(X_k)\Delta^{(2)} B_k^H| \cdot V_p(X;[t_k,t_{k+1}]) V_p(B^H;[t_k,t_{k+1}])
\end{aligned}$$

$$+ 2C_{p,p}|g'|_\infty \sum_{k=1}^{n-1} \left|g(X_k)\Delta^{(2)} B_k^H\right| \cdot V_p(X;[t_{k-1},t_k])V_p(B^H;[t_{k-1},t_k])$$

which, coupled with the inequality (1.7), yields

$$\begin{aligned}
|\widetilde{S_2}| \leqslant & 2C_{p,p}^2 |g'|_\infty^2 \max_{0\leqslant k\leqslant n-1} \left[V_p(X;[t_k,t_{k+1}])V_p(B^H;[t_k,t_{k+1}])\right] \\
& \times V_p(X;[0,T])V_p(B^H;[0,T]) \\
& + 4C_{p,p}|g'|_\infty \max_{1\leqslant k\leqslant n-1} |g(X_k)\Delta^{(2)}B_k^H| V_p(X;[0,T])V_p(B^H;[0,T]) \\
\leqslant & 2C_{p,p}^2|g'|_\infty^2 \max_{0\leqslant k\leqslant n-1} \left[V_p(B^H;[t_k,t_{k+1}])\right] V_p^2(X;[0,T])V_p(B^H;[0,T]) \\
& + 8C_{p,p}|g'|_\infty \max_{1\leqslant k\leqslant n} |\Delta B_k^H| \left[|g'|_\infty V_p(X;[0,T]) + |g(\xi)|\right] \\
& \times V_p(X;[0,T])V_p(B^H;[0,T])
\end{aligned}$$

since for all $0 \leqslant k \leqslant n$

$$|g(X_k)| \leqslant |g'|_\infty V_p(X;[0,T]) + |g(\xi)|.$$

$|S_{12}|$ can be rewritten as

$$\begin{aligned}
|S_{12}| = & 2\sum_{k=1}^{n-1} \left| \int_{t_k}^{t_{k+1}} g(X_s)\,dB_s^H - g(X_k)\Delta B_k^H \right. \\
& \left. - \int_{t_{k-1}}^{t_k} g(X_s)\,dB_s^H + g(X_k)\Delta B_k^H \right| \\
& \times \left| \int_{t_k}^{t_{k+1}} f(X_s)\,ds - \int_{t_{k-1}}^{t_k} f(X_s)\,ds \right|.
\end{aligned}$$

From the Love-Young inequality (1.4) it follows that

$$\begin{aligned}
|S_{12}| \leqslant & 2C_{p,p} \sum_{k=1}^{n-1} \Bigg\{ \Big[V_p(g(X);[t_k,t_{k+1}])V_p(B^H;[t_k,t_{k+1}]) \\
& + V_p(g(X);[t_{k-1},t_k])V_p(B^H;[t_{k-1},t_k]) \Big] \\
& \times \left[\int_{t_k}^{t_{k+1}} |f(X_s)|\,ds + \int_{t_{k-1}}^{t_k} |f(X_s)|\,ds \right] \Bigg\} \\
\leqslant & 4C_{p,p} \max_{1\leqslant k\leqslant n} \left[V_p(g(X);[t_{k-1},t_k])V_p(B^H;[t_{k-1},t_k])\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{k=1}^{n-1} \left[\int_{t_k}^{t_{k+1}} |f(X_s)|\, ds + \int_{t_{k-1}}^{t_k} |f(X_s)|\, ds \right] \\
&\leqslant 8C_{p,p}|g'|_\infty V_p(X;[0,T]) \max_{1\leqslant k\leqslant n} \left[V_p(B^H;[t_{k-1},t_k])\right] \\
&\times \int_0^T |f(X_s)|\, ds\,.
\end{aligned}
$$

Additionally,

$$
|f(X_s)| \leqslant |f(X_s)-f(\xi)| + |f(\xi)| \leqslant LV_p(X;[0,T]) + |f(\xi)|\,,
$$

where L is the Lipschitz constant of the function f. Therefore,

$$
\begin{aligned}
|S_{12}| \leqslant 8C_{p,p}|g'|_\infty V_p(X;[0,T]) &\max_{1\leqslant k\leqslant n} \left[V_p(B^H;[t_{k-1},t_k])\right] \\
&\times \left[LV_p(X;[0,T]) + |f(\xi)|\right] \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \sum_{k=1}^{n-1} \left(\int_{t_k}^{t_{k+1}} f(X_s)\, ds - \int_{t_{k-1}}^{t_k} f(X_s)\, ds \right)^2 \\
&\leqslant 2\sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} f(X_s)\, ds \right)^2 \leqslant 2n^{-1} \int_0^T f^2(X_s)\, ds \\
&\leqslant 2Tn^{-1}\left[|f(\xi)| + LV_p(X;[0,T])\right]^2.
\end{aligned}
$$

Note that for p such that $H - 1/p < 1 - H$ it follows that

$$
\begin{aligned}
n^{2H-1} &\max\left\{|\Delta B_k^H|, V_p(B^H;[t_k,t_{k+1}])\right\} \\
&\leqslant L_T^{H,1/p} T^{1/p} n^{2H-1-1/p} \xrightarrow{\text{a.s.}} 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Such a value of p will always exist. Therefore it follows that

$$
n^{2H-1}|S_1 + \tilde{S_2} + S_{12}| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n\to\infty.
$$

Consequently, the theorem will be proved if the convergence

$$
n^{2H-1}\sum_{k=1}^{n-1} g^2(X_k)(\Delta^{(2)}B_k^H)^2 \xrightarrow{\text{a.s.}} (4-2^{2H}) \int_0^T g^2(X_t)\, dt
$$

is obtained. To do so, the Helly-Bray theorem is applied.

2.5 Theorem. *Let the functions F_n ($n = 1, 2, \dots$) be non-decreasing and uniformly bounded. If the sequence F_n converges to F in its continuity points and*

$$F_n(-\infty) \to F(-\infty), \quad F_n(\infty) \to F(\infty),$$

then for every continuous bounded function g

$$\int_{-\infty}^{\infty} g(x)\, dF_n(x) \to \int_{-\infty}^{\infty} g(x)\, dF(x)\,.$$

Let

$$V_n^{(2)}(B^H, 2)_t = \sum_{k=1}^{[nt/T]-1} (\Delta^{(2)} B_k^H)^2\,, \quad t \in [0, T]$$

and $S_n(t) = n^{2H-1} V_n^{(2)}(B^H, 2)(t)$. Then

$$n^{2H-1} \sum_{k=1}^{n-1} g^2(X_k)(\Delta^{(2)} B_k^H)^2 = \int_0^T g^2(X_t)\, dS_n(t).$$

It is known (see, e.g., Bégyn (2006) 122p.) that

$$n^{2H-1} V_n^2(B^H, 2)_t \xrightarrow{\text{a.s.}} (4 - 2^{2H})t\,.$$

Since the function S_t^n is non-decreasing, it follows that (see Lemma 1 in McLeish (1978))

$$\sup_{t \leqslant T} |S_t^n - t| \xrightarrow{\text{a.s.}} 0 \qquad \text{as } n \to \infty\,.$$

The function S_t^n is non-decreasing and uniformly bounded for every ω. Therefore the Helly-Bray theorem yields

$$\int_0^T g^2(X_t)\, dS_n(t) \xrightarrow{\text{a.s.}} (4 - 2^{2H}) \int_0^T g^2(X_t)\, dt \quad \text{as} \quad n \to \infty$$

which completes the proof.

Proof of Theorem 2.4. The estimator $\widehat{H}_{dv2}^n$ can be rewritten as

$$\widehat{H}_{dv2}^n = \frac{1}{2} - \frac{1}{2\ln 2}\Big[(2H-1)\ln\frac{1}{2} + \ln\frac{(2n)^{2H-1} V_{2n}(X,2)}{(n)^{2H-1} V_n(X,2)}\Big]$$

$$=H - \frac{1}{2\ln 2}\ln\frac{(2n)^{2H-1}V_{2n}(X,2)}{(n)^{2H-1}V_n(X,2)},$$

which, coupled with the results of Theorem 2.3 yields the convergence

$$\widehat{H}^n_{dv2} - H \xrightarrow{\mathbf{a.s.}} 0.$$

2.2. Irregular subdivisions

2.6 Theorem. *Let f be a Lipschitz-continuos function and let $g \in \mathcal{C}^{1+\alpha}$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of the interval $[0, T]$ such that*

$$m_n^{2-2H} \overset{\mathbf{n}\to\infty}{=} o(1/\ln n) \quad \text{and} \quad m_n \overset{\mathbf{n}\to\infty}{=} \mathcal{O}(p_n).$$

Then

$$V^{(1)}_{\pi_n}(X,2) \xrightarrow{\mathbf{a.s.}} \int_0^T g^2(X_t)\,dt \qquad \text{as } m_n \to 0,$$

where X is the solution of (1.1).

Let $(\pi_n)_{n\geqslant 1}$ be a sequence of partitions of $[0, T]$ such that $0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T$ for all $n \geqslant 1$. Assume that we have two sequences of partitions $(\pi_{i(n)})_{n\geqslant 1}$ and $(\pi_{j(n)})_{n\geqslant 1}$ of $[0, T]$ such that $\pi_{i(n)} \subset \pi_{j(n)} \subseteq \pi_n$, $i(n) < j(n) \leqslant N_n$, for all $n \in \mathbb{N}$, where $\pi_{i(n)} = \{0 = t_0^n < t_{i(1)}^n < t_{i(2)}^n < \cdots < t_{i(n)}^n = T\}$ and $\pi_{j(n)} = \{0 = t_0^n < t_{j(1)}^n < t_{j(2)}^n < \cdots < t_{j(n)}^n = T\}$. Define

$$\widetilde{H}^n_{dv1} := \frac{1}{2} - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})}\ln\frac{V^{(1)}_{j(n)}(X,2)}{V^{(1)}_{i(n)}(X,2)}, \quad V^{(1)}_{i(n)}(X,2) = \sum_{k=1}^{i(n)} (\Delta X_k^n)^2,$$

where

$$\Delta t_k^n = t_{i(k)}^n - t_{i(k-1)}^n, \qquad m_{i(n)} = \max_{1\leqslant k\leqslant i(n)} \Delta t_k^n, \qquad p_{i(n)} = \min_{1\leqslant k\leqslant i(n)} \Delta t_k^n.$$

2.7 Theorem. *Assume that conditions of Theorem* 2.6 *are satisfied. If the sequences of partitions $(\pi_{i(n)})$ and $(\pi_{j(n)})$, $i(n) < j(n)$, are regular or such that $\ln(p_{i(n)}/p_{j(n)}) \to \infty$ as $n \to \infty$, then*

$$\widetilde{H}^n_{dv1} \longrightarrow H \quad \textit{a.s.} \qquad \textit{as } n \to \infty.$$

To study the almost sure convergence of the second order quadratic variations of X additional assumptions on the sequence $(\pi_n)_{n\in\mathbb{N}}$ are required.

2.1 Definition. (see Bégyn (2006)) *Let $(\ell_k)_{k\geqslant 1}$ be a sequence of real numbers in the interval $(0, \infty)$. We say that $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of subdivisions with asymptotic ratios $(\ell_k)_{k\geqslant 1}$ if it satisfies the following assumptions:*

- $m_n \overset{\mathbf{n}\to\infty}{=} \mathcal{O}(p_n)$;
- $\lim_{n\to\infty} \sup_{1\leqslant k\leqslant N_n} \left| \frac{\Delta t^n_{k-1}}{\Delta t^n_k} - \ell_k \right| = 0.$

The set $\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_k, \dots\}$ will be called the range of the asymptotic ratios of the sequence $(\pi_n)_{n\in\mathbb{N}}$.

It is clear that if the sequence $(\pi_n)_{n\in\mathbb{N}}$ is regular, then it is a sequence with asymptotic ratios $\ell_k = 1$ for all $k \geqslant 1$.

2.2 Definition. (see Bégyn (2006)) *The function $g : (0, \infty) \to \mathbb{R}$ is invariant on $\mathcal{L}$ if for all $\ell, \hat{\ell} \in \mathcal{L}$, $g(\ell) = g(\hat{\ell})$.*

For example, let $\mathcal{L} = \{\alpha, \alpha^{-1}\}$ be the set containing two real positive numbers and let

$$h(\lambda) = \frac{1 + \lambda^{2H-1} - (1+\lambda)^{2H-1}}{\lambda^{H-1/2}} .$$

The function h is invariant on $\mathcal{L}$.

2.8 Theorem. *Let f be a Lipschitz-continuos function and let $g \in \mathcal{C}^{1+\alpha}$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions with asymptotic ratios $(\ell_k)_{k\geqslant 1}$ and range of the asymptotic ratios $\mathcal{L}$. Assume that the lower mesh of the subdivisions π_n satisfy $p_n \overset{\mathbf{n}\to\infty}{=} o(1/\ln n)$ and*

$$h(\lambda) = \frac{1 + \lambda^{2H-1} - (1+\lambda)^{2H-1}}{\lambda^{H-1/2}} .$$

Let X be the solution of (1.1). If the function h is invariant on $\mathcal{L}$ or the sequence of functions $\ell_n(t)$ converges uniformly to $\ell(t)$ on the interval $[0, T]$, where

$$\ell_n(t) = \sum_{k=1}^{N_n-1} \ell_k \mathbf{1}_{[t^n_k, t^n_{k+1})}(t),$$

then

$$\lim_{n\to\infty} V_{\pi_n}^{(2)}(X,2) = 2\int_0^T g^2(X_t)h(\ell(t))\,dt.$$

Proof of Theorem 2.6. The p-variation boundedness of almost all paths of solution of the equation (1.1) implies the existence of the integrals $\int_0^T f^2(X_t)\,dt$ and $\int_0^T g^2(X_t)\,dt$.

Set $X_k^n := X(t_k^n)$. For simplicity, the index H for B will be omitted in the sequel. Note that

$$\begin{aligned}(\Delta X_k^n)^2 =& \Big(\int_{t_{k-1}^n}^{t_k^n} f(X_s)\,ds\Big)^2 + 2\int_{t_{k-1}^n}^{t_k^n} f(X_s)\,ds\cdot\int_{t_k^n}^{t_k^n} g(X_s)\,dB_s\\ &+\Big(\int_{t_{k-1}^n}^{t_k^n}[g(X_s)-g(X_k^n)]dB_s\Big)^2 + g^2(X_k^n)(\Delta B_k^n)^2\\ &+2g(X_k^n)\Delta B_k^n\Big(\int_{t_{k-1}^n}^{t_k^n}[g(X_s)-g(X_k^n)]dB_s\Big)\\ =&\sum_{j=1}^5 I_{n,k}^{(j)}\,.\end{aligned}$$

It'll be proved that

$$\sum_{\substack{j=1\\ j\neq 4}}^5 I_{n,k}^{(j)} \xrightarrow{\textbf{a.s.}} 0 \quad \text{as} \quad m_n\to 0.$$

Note that

$$|I_{n,k}^{(1)}| \leqslant \Delta t_k^n \int_{t_{k-1}^n}^{t_k^n} f^2(X_s)\,ds.$$

By Love-Young inequality (1.4) for all $p>1/H$

$$\begin{aligned}|I_{n,k}^{(2)}| \leqslant& 2\Big|\int_{t_{k-1}^n}^{t_k^n} f(X_s)\,ds\Big|\cdot|g(X_k)\Delta B_k^n|\\ &+2\Big|\int_{t_{k-1}^n}^{t_k^n} f(X_s)\,ds\Big|\cdot\Big|\int_{t_{k-1}^n}^{t_k^n}[g(X_s)-g(X_k^n)]dB_s\Big|\\ \leqslant& 2|\Delta B_k^n|\sup_{t\leqslant T}|g(X_t)|\int_{t_{k-1}^n}^{t_k^n}|f(X_s)|\,ds\end{aligned}$$

$$
\begin{aligned}
&\quad + 2C_{p,p}|g'|_\infty V_p(X;[t_k^n;t_{k+1}^n])V_p(B;[t_k^n;t_{k+1}^n]) \int_{t_{k-1}^n}^{t_k^n} |f(X_s)|\, ds,\\
|I_{n,k}^{(3)}| &\leqslant C_{p,p}^2 |g'|_\infty^2 V_p^2(X;[t_k^n;t_{k+1}^n])V_p^2(B;[t_k^n;t_{k+1}^n]),\\
|I_{n,k}^{(5)}| &\leqslant 2C_{p,p}|g'|_\infty |\Delta B_k^n| \sup_{t\leqslant T}|g(X_t)| V_p(X;[t_k^n;t_{k+1}^n])V_p(B;[t_k^n;t_{k+1}^n]).
\end{aligned}
$$

It is evident that

$$
\begin{aligned}
&\sum_{k=1}^{N_n} (\Delta t_k)^{1-2H} |I_{n,k}^{(1)}| \leqslant m_n^{2-2H} \int_0^T f^2(X_s)\, ds\,,\\
&\sum_{k=1}^{N_n} (\Delta t_k)^{1-2H} |I_{n,k}^{(2)}| \leqslant 2 \max_{1\leqslant k\leqslant N_n} \frac{|\Delta B_k^n|}{(\Delta t_k)^{2H-1}} \sup_{t\leqslant T}|g(X_t)| \int_0^T |f(X_s)|\, ds\\
&\quad + 2C_{p,p}|g'|_\infty \max_{1\leqslant k\leqslant N_n-1} \frac{V_p(B;[t_k,t_{k+1}])}{(\Delta t_k)^{2H-1}} V_p(X;[0,T]) \int_0^T |f(X_s)|\, ds\,.
\end{aligned}
$$

By using the inequality (1.7) the remaining two terms are estimated as

$$
\begin{aligned}
&\sum_{k=1}^{N_n} (\Delta t_k)^{1-2H} |I_{n,k}^{(3)}|\\
&\quad \leqslant C_{p,p}^2 |g'|_\infty^2 \max_{1\leqslant k\leqslant N_n-1} \frac{V_p(B;[t_k,t_{k+1}])}{(\Delta t_k)^{2H-1}} V_p^2(X;[0,T])V_p(B;[0,T]),\\
&\sum_{k=1}^{N_n} (\Delta t_k)^{1-2H} |I_{n,k}^{(5)}|\\
&\quad \leqslant 2C_{p,p}|g'|_\infty \max_{1\leqslant k\leqslant N_n} \frac{|\Delta B_k^n|}{(\Delta t_k)^{2H-1}} \sup_{t\leqslant T}|g(X_t)| V_p(X;[0,T])V_p(B;[0,T]).
\end{aligned}
$$

By (1.8) it follows that

$$
\max_{1\leqslant k\leqslant N_n} |\Delta B_k^n| \leqslant L_T^{H,1/p} m_n^{1/p}, \qquad \max_{1\leqslant k\leqslant N_n} \left[V_p(B;[t_{k-1}^n;t_k^n])\right] \leqslant L_T^{H,1/p} m_n^{1/p}.
$$

All the inequalities obtained above are correct for every $p > 1/H$. Thus there can always be chosen such a p that $1/p + 1 - 2H > 0$. For this value of p

$$
V_{\pi_n}^{(1)}(X,2) - \sum_{k=1}^{N_n} g^2(X_k) \frac{\left(\Delta B_k^n\right)^2}{(\Delta t_k^n)^{2H-1}} \xrightarrow{\mathbf{a.s.}} 0
$$

as $m_n \to 0$. Consequently, the theorem will be proved if the convergence

$$\sum_{k=1}^{N_n} g^2(X_k^n) \frac{(\Delta B_k^n)^2}{(\Delta t_k^n)^{2H-1}} \xrightarrow{\text{a.s.}} \int_0^T g^2(X_t)\, dt.$$

is obtained. Set

$$S_t^n = \sum_{k=1}^{r^n(t)} \frac{(\Delta B_k^n)^2}{(\Delta t_k^n)^{2H-1}}\,, \quad t \in [0, T],$$

where $r^n(t) = \max\{k\colon\ t_k^n \leqslant t\}$. Then

$$\sum_{k=1}^{N_n} g^2(X_k^n) \frac{(\Delta B_k^n)^2}{(\Delta t_k^n)^{2H-1}} = \int_0^T g^2(X_t)\, dS_t^n.$$

It is known (see Gine, Klein (1975)) that $S_t^n \xrightarrow{\text{a.s.}} t$ if

$$m_n^{2-2H} \overset{\mathbf{n\to\infty}}{=} o(1/\ln n) \quad \text{and} \quad m_n \overset{\mathbf{n\to\infty}}{=} \mathcal{O}(p_n).$$

The function S_t^n is non-decreasing and uniformly bounded. Consequently, the Helly-Bray theorem implies that

$$\int_0^T g^2(X_t)\, dS_t^n \xrightarrow{\text{a.s.}} \int_0^T g^2(X_t)\, dt \qquad \text{as} \quad n \to \infty\,.$$

This completes the proof of the theorem.

Proof of Theorem 2.7. Note that

$$\frac{m_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{p_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)} \leqslant \frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)} \leqslant \frac{p_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{m_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)}\,. \tag{2.1}$$

It is evident that

$$\begin{aligned}\widehat{H}_n^{(1)} =& \frac{1}{2} - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \\ & \times \left[(2H-1)\ln(p_{j(n)}/m_{i(n)}) + \ln \frac{p_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{m_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)} \right]\end{aligned}$$

$$
\begin{aligned}
&= H - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \ln \frac{p_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{m_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)} \\
&= H - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \\
&\quad \times \left[\ln \frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)} + \ln \left(\frac{p_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{m_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)} \Big/ \frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)} \right) \right].
\end{aligned}
$$

From the inequality (2.9) it follows that

$$
\ln \left(\frac{p_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{m_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)} \Big/ \frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)} \right) \geqslant 0.
$$

Also $\ln(m_{i(n)}/p_{j(n)}) > 0$. Thus

$$
\widehat{H}_n^{(1)} \leqslant H - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \ln \frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)}.
$$

Further,

$$
\begin{aligned}
\widehat{H}_n^{(1)} &= \frac{1}{2} - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \\
&\quad \times \left[(2H-1)\ln(m_{j(n)}/p_{i(n)}) + \ln \frac{m_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{p_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)} \right] \\
&= \frac{1}{2} - \Big(H - \frac{1}{2} \Big) \frac{\ln(m_{j(n)}/p_{i(n)})}{\ln(m_{i(n)}/p_{j(n)})} \\
&\quad - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \ln \frac{m_{j(n)}^{1-2H} V_{j(n)}^{(1)}(X,2)}{p_{i(n)}^{1-2H} V_{i(n)}^{(1)}(X,2)} \\
&= H + \Big(H - \frac{1}{2} \Big) \frac{\ln(p_{i(n)}/m_{j(n)}) - \ln(m_{i(n)}/p_{j(n)})}{\ln(m_{i(n)}/p_{j(n)})} \\
&\quad - \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \Big[\ln \frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)}
\end{aligned}
$$

$$+\ln\Big(\frac{m_{j(n)}^{1-2H}V_{j(n)}^{(1)}(X,2)}{p_{i(n)}^{1-2H}V_{i(n)}^{(1)}(X,2)}\Big/\frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)}\Big)\Big].$$

Again, from (2.9) it follows that

$$\ln\Big(\frac{m_{j(n)}^{1-2H}V_{j(n)}^{(1)}(X,2)}{p_{i(n)}^{1-2H}V_{i(n)}^{(1)}(X,2)}\Big/\frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)}\Big)\leqslant 0$$

which implies

$$\begin{aligned}\widehat{H}_n^{(1)} \geqslant & H+\Big(H-\frac{1}{2}\Big)\frac{\ln(p_{i(n)}/m_{i(n)})+\ln(p_{j(n)}/m_{j(n)})}{\ln(m_{i(n)}/p_{j(n)})}\\ & -\frac{1}{2\ln(m_{i(n)}/p_{j(n)})}\ \ln\frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)}\,. \end{aligned}\tag{2.2}$$

If the sequences of partitions $(\pi_{i(n)})$ and $(\pi_{j(n)})$, $i(n) < j(n)$, are regular then the second term in the inequality (2.2) is equal to 0. If $\ln(p_{i(n)}/p_{j(n)}) \to \infty$, $n \to \infty$, then the second term in the inequality (2.2) converges to 0. By Theorem 2.6 we get that

$$\frac{1}{2\ln(m_{i(n)}/p_{j(n)})}\ \ln\frac{V_{\pi_{j(n)}}^{(1)}(X,2)}{V_{\pi_{i(n)}}^{(1)}(X,2)}\longrightarrow 0\quad \text{a.s.}\qquad \text{as } n\to\infty.$$

Therefore the convergence $\widehat{H}_n^{(1)} \longrightarrow H$ a.s. as $n \to \infty$ holds.

Proof of Theorem 2.8. It is obvious that the square of the second order increments can be rewritten as

$$\begin{aligned}(\Delta_{ir}^{(2)}X_k^n)^2 = & \Big(\Delta t_k^n\int_{t_k^n}^{t_{k+1}} f(X_s)\,ds-\Delta t_{k+1}^n\int_{t_{k-1}^n}^{t_k^n} f(X_s)\,ds\Big)^2\\ & +2\Big(\Delta t_k^n\int_{t_k^n}^{t_{k+1}^n}\big[g(X_s)-g(X_k^n)\big]dB_s\\ & \quad -\Delta t_{k+1}^n\int_{t_{k-1}^n}^{t_k^n}\big[g(X_s)-g(X_k^n)\big]dB_s\Big)\end{aligned}$$

$$\begin{aligned}
&\times \left(\Delta t_k^n \int_{t_k^n}^{t_{k+1}^n} f(X_s)\,ds - \Delta t_{k+1}^n \int_{t_{k-1}^n}^{t_k^n} f(X_s)\,ds\right) \\
&+ 2g(X_k^n)\,\Delta_{ir}^{(2)} B_k^n \left(\Delta t_k^n \int_{t_k^n}^{t_{k+1}^n} f(X_s)\,ds - \Delta t_{k+1}^n \int_{t_{k-1}^n}^{t_k^n} f(X_s)\,ds\right) \\
&+ \left(\Delta t_k^n \int_{t_k^n}^{t_{k+1}^n} [g(X_s) - g(X_k^n)]\,dB_s \right. \\
&\qquad \left. - \Delta t_{k+1}^n \int_{t_{k-1}^n}^{t_k^n} [g(X_s) - g(X_k^n)]\,dB_s \right)^2 \\
&+ 2g(X_k^n)\,\Delta_{ir}^{(2)} B_k^n \left(\Delta t_k^n \int_{t_k^n}^{t_{k+1}^n} [g(X_s) - g(X_k^n)]\,dB_s \right. \\
&\qquad \left. - \Delta t_{k+1}^n \int_{t_{k-1}^n}^{t_k^n} [g(X_s) - g(X_k^n)]\,dB_s \right) \\
&+ g^2(X_k^n)(\Delta_{ir}^{(2)} B_k^n)^2 = \sum_{i=1}^{6} I_{n,k}^{(i)}.
\end{aligned}$$

The Love-Young inequality and simple calculations yield

$$\begin{aligned}
I_{n,k}^{(1)} &\leqslant 2m_n^3 \int_{t_k^n}^{t_{k+1}^n} f^2(X_s)\,ds + 2m_n^3 \int_{t_{k-1}^n}^{t_k^n} f^2(X_s)\,ds, \\
|I_{n,k}^{(2)}| &\leqslant 4C_{p,p}|g'|_\infty m_n^2 \max_{1\leqslant k\leqslant N_n} [V_p(B;[t_{k-1}^n, t_k^n])]\, V_p(X;[0,T]) \\
&\qquad \times \left[\int_{t_k^n}^{t_{k+1}^n} |f(X_s)|\,ds + \int_{t_{k-1}^n}^{t_k^n} |f(X_s)|\,ds\right], \\
|I_{n,k}^{(3)}| &\leqslant 2m_n \max_{1\leqslant k\leqslant N_n-1} |\Delta_{ir}^{(2)} B_k| \sup_{t\leqslant T} |g(X_t)| \\
&\qquad \times \left[\int_{t_k^n}^{t_{k+1}^n} |f(X_s)|\,ds + \int_{t_{k-1}^n}^{t_k^n} |f(X_s)|\,ds\right], \\
|I_{n,k}^{(4)}| &\leqslant 2C_{p,p}^2 |g'|_\infty^2 m_n^2 [V_p^2(X;[t_k^n, t_{k+1}^n]) V_p^2(B;[t_k^n, t_{k+1}^n]) \\
&\qquad + V_p^2(X;[t_{k-1}^n, t_k^n]) V_p^2(B;[t_{k-1}^n, t_k^n])], \\
|I_{n,k}^{(5)}| &\leqslant 2C_{p,p}|g'|_\infty m_n \max_{1\leqslant k\leqslant N_n-1} |\Delta_{ir}^{(2)} B_k^n| \sup_{t\leqslant T} |g(X_t)| \\
&\qquad \times \{V_p(X;[t_k, t_{k+1}]) V_p(B;[t_k^n, t_{k+1}^n]) \\
&\qquad\qquad + V_p(X;[t_{k-1}^n, t_k^n]) V_p(B;[t_{k-1}^n, t_k^n])\}.
\end{aligned}$$

Set

$$\mu_k^n = 2\,\frac{1}{(\Delta t_k^n)^{1/2+H}(\Delta t_{k+1}^n)^{1/2+H}(\Delta t_k^n + \Delta t_{k+1}^n)}\,.$$

Note that $\mu_k^n \leqslant \frac{1}{p_n^{2+2H}}$ and

$$\max_{1\leqslant k\leqslant N_n-1} |\Delta_{ir}^{(2)} B_k^n| \leqslant 2m_n \max_{1\leqslant k\leqslant N_n} |\Delta B_k^n| \leqslant 2m_n^{1+1/p} L_T^{H,1/p}\,.$$

Thus

$$\begin{aligned}
\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_k \cdot I_{n,k}^{(1)} \leqslant& 4\,\frac{m_n^4}{p_n^{2+2H}} \int_0^T f^2(X_s)\,ds,\\
\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_k \cdot I_{n,k}^{(2)} \leqslant& 8C_{p,p}|g'|_\infty \frac{m_n^3}{p_n^{2+2H}} \max_{1\leqslant k\leqslant N_n} V_p(B;[t_{k-1}^n, t_k^n])\\
&\times V_p(X;[0,T]) \int_0^T |f(X_s)|\,ds,
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_k \cdot I_{n,k}^{(3)} \leqslant& 8\frac{m_n^3}{p_n^{2+2H}} \max_{1\leqslant k\leqslant N_n} |\Delta B_k^n| \sup_{t\leqslant T} |g(X_t)| \int_0^T |f(X_s)|\,ds,\\
\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_k \cdot I_{n,k}^{(4)} \leqslant& 4C_{p,p}^2 |g'|_\infty^2 \frac{m_n^3}{p_n^{2+2H}} \max_{1\leqslant k\leqslant N_n} V_p(B;[t_{k-1}^n, t_k^n]) V_p^2(X;[0,T])\\
&\times V_p(B;[0,T]),\\
\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_k^n \cdot I_{n,k}^{(5)} \leqslant& 8C_{p,p}|g'|_\infty \frac{m_n^3}{p_n^{2+2H}} \max_{1\leqslant k\leqslant N_n} |\Delta B_k^n| \sup_{t\leqslant T} |g(X_t)|\\
&\times V_p(X;[0,T]) V_p(B^H;[0,T]).
\end{aligned}$$

The inequalities obtained above are valid for all $p > 1/H$. By Definition 2.1

$$\frac{m_n^{3+1/p}}{p_n^{2+2H}} \longrightarrow 0, \qquad \frac{m_n^4}{p_n^{2+2H}} \longrightarrow 0, \qquad \frac{m_n^{3+2/p}}{p_n^{2+2H}} \longrightarrow 0 \qquad \text{as } n \to \infty,$$

if $3 + 1/p - 2 - 2H > 0$ and $3 + 2/p - 2 - 2H > 0$. Such a value of p can always be chosen. Thus

$$\sum_{j=1}^{5}\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_k^n\cdot I_{n,k}^{(j)}\xrightarrow{\mathbf{a.s.}}0\qquad\text{as } n\to\infty$$

and

$$V_{\pi_n}^{(2)}(X,2)-\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_k^n g^2(X_k^n)(\Delta_{ir}^{(2)}B_k^n)^2\xrightarrow{\mathbf{a.s.}}0\qquad\text{as } n\to\infty.$$

Denote $r^n(t)=\max\{k\colon\ t_k^n\leqslant t\}$ and assume that

$$\begin{aligned}V_{\pi_n}^{(2)}(X,2)_t&=2\sum_{k=1}^{r^n(t)-1}\frac{\Delta t_{k+1}^n(\Delta_{ir}^{(2)}X_k^n)^2}{(\Delta t_k^n)^{1/2+H}(\Delta t_{k+1}^n)^{1/2+H}(\Delta t_k^n+\Delta t_{k+1}^n)}\\&=:2\sum_{k=1}^{r^n(t)-1}\mu_k^n\Delta t_{k+1}^n(\Delta_{ir}^{(2)}X_k^n)^2\,.\end{aligned}$$

From the results obtained in Bégyn (2006) it is easy to see that for every $t\in[0,T]$

$$V_{\pi_n}^{(2)}(B^H,2)_t-\mathbf{E}V_{\pi_n}^{(2)}(B^H,2)_t\xrightarrow{\mathbf{a.s.}}0\qquad\text{as } n\to\infty\,.$$

Let's show that

$$\mathbf{E}V_{\pi_n}^{(2)}(B^H,2)_t=2\sum_{k=1}^{r^n(t)-1}h(\ell_k)\Delta t_{k+1}^n\,.$$

Obviously,

$$\mathbf{E}V_{\pi_n}^{(2)}(X,2)_t=2\sum_{k=1}^{r^n(t)-1}\mu_k^n\Delta t_{k+1}^n\mathbf{E}(\Delta_{ir}^{(2)}X_k^n)^2\,,$$

and simple calculations yield

$$\begin{aligned}\mathbf{E}(\Delta_{ir}^{(2)}X_k^n)^2=&\,\mathbf{E}\left(\Delta t_k\Delta B_{k+1}^H-\Delta t_{k+1}\Delta B_k^H\right)^2=(\Delta t_k)^2\cdot(\Delta t_{k+1})^{2H}\\&-2\Delta t_k\Delta t_{k+1}\mathbf{E}\Delta B_k^H\Delta B_{k+1}^H+(\Delta t_{k+1})^2\cdot(\Delta t_k)^{2H}\\=&\,(\Delta t_k)^2\cdot(\Delta t_{k+1})^{2H}+(\Delta t_{k+1})^2\cdot(\Delta t_k)^{2H}\end{aligned}$$

$$\begin{aligned}&- \Delta t_k \Delta t_{k+1} \left[(\Delta t_{k+1} + \Delta t_k)^{2H} - (\Delta t_{k+1})^{2H} - (\Delta t_k)^{2H} \right] \\ =& \Delta t_k \Delta t_{k+1} \left(\Delta t_k + \Delta t_{k+1} \right) \Big[(\Delta t_{k+1})^{2H-1} + (\Delta t_k)^{2H-1} \\ &\quad - (\Delta t_{k+1} + \Delta t_k)^{2H-1} \Big] .\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbf{E} V^{(2)}_{\pi_n}(X,2)_t \\ &\quad = 2 \sum_{k=1}^{r^n(t)-1} \Delta t_{k+1} \frac{(\Delta t_{k+1})^{2H-1} - (\Delta t_{k+1} + \Delta t_k)^{2H-1} + (\Delta t_k)^{2H-1}}{(\Delta t^n_k)^{H-1/2} \cdot \left(\Delta t^n_{k+1}\right)^{H-1/2}} \\ &\quad = 2 \sum_{k=1}^{r^n(t)-1} \Delta t_{k+1} \frac{1 - (\ell_k + 1)^{2H-1} + \ell_k^{2H-1}}{\ell_k^{H-1/2}} = 2 \sum_{k=1}^{r^n(t)-1} h(\ell_k) \Delta t^n_{k+1} .\end{aligned}$$

Further it'll be proved that

$$\sup_{t \leqslant T} |S^n_t| \xrightarrow{\text{a.s.}} 0 \qquad \text{as } n \to \infty, \quad \text{where} \tag{2.3}$$

$$S^n_t = 2 \sum_{k=1}^{r^n(t)-1} \left[\mu^n_k \Delta t^n_{k+1} (\Delta^{(2)}_{ir} B^H_k)^2 - h(\ell_k) \Delta t^n_{k+1} \right], \quad t \in [0,T].$$

Let (s^m_j), $0 \leqslant j \leqslant m$, $m \geqslant 1$, be a sequence of partitions of the interval $[0,\, T]$, $0 = s^m_0 < s^m_1 < \ldots < s^m_m = T$, such that $\max_{1 \leqslant j \leqslant m}(s^m_j - s^m_{j-1}) \to 0$ as $m \to \infty$. To prove (2.3) it is suffices to show that for every such sequence (s^m_j) (see Lemma 5 in Liptser, Shiryaev (1989) p. 556-557)

$$\lim_{m \to \infty} \limsup_{n \to \infty} \max_{1 \leqslant j \leqslant m-1} \left| \sum_{k=1}^{r^n(s^m_{j+1})-1} h(\ell_k) \Delta t^n_{k+1} - \sum_{k=1}^{r^n(s^m_j)-1} h(\ell_k) \Delta t^n_{k+1} \right| = 0.$$

Note that the function h is continuous and bounded, i. e. $0 \leqslant h(\lambda) \leqslant 1$ for $\lambda \geqslant 0$. Thus

$$\begin{aligned}&\max_{1 \leqslant j \leqslant m-1} \left| \sum_{k=1}^{r^n(s^m_{j+1})-1} h(\ell_k) \Delta t^n_{k+1} - \sum_{k=1}^{r^n(s^m_j)-1} h(\ell_k) \Delta t^n_{k+1} \right| \\ &\qquad \leqslant \max_{1 \leqslant j \leqslant m-1} \left| \rho^n(s^m_{j+1}) - \rho^n(s^m_j) \right| ,\end{aligned}$$

where $\rho^n(t) = \max\{t_k^n\colon\ t_k^n \leqslant t\}$, and for sufficiently large n the value of m can be chosen such that the right side of inequality is a diminutive value. Therefore S_t^n is uniformly bounded for almost all ω and by the Helly-Bray theorem

$$\sum_{k=1}^{N_n-1} g^2(X_k^n)\,[\mu_k^n \Delta t_k^n (\Delta_{ir}^{(2)} B_k^n)^2 - 2h(\ell_k)\Delta t_{k+1}^n] \\ = \int_0^T g^2(X_t)\, dS_t^n \xrightarrow{\mathbf{a.s.}} 0 \quad \text{as} \qquad n \to \infty\,.$$

Then

$$\sum_{k=1}^{N_n-1} g^2(X_k^n) h(\ell_k) \Delta t_{k+1}^n = \int_0^T g^2(X_t^{\pi_n}) h\big(\ell_n(t)\big)\, dt$$

and

$$\int_0^T \big|g^2(X_t) h\big(\ell(t)\big) - g^2(X_t^{\pi_n}) h\big(\ell_n(t)\big)\big|\, dt \\ \leqslant T \sup_{t\leqslant T} |g^2(X_t)| \sup_{t\leqslant T} |h(\ell_n(t)) - h(\ell(t))| \\ + T \sup_{t\leqslant T} \big|g^2(X_t) - g^2(X_t^{\pi_n})\big| \longrightarrow 0 \qquad \text{as} \quad n \to \infty$$

which completes the proof.

2.3. The Milstein approximation

For a process $\{X_t\}$, its Milstein approximation at the points t_k^n, $k = 1, \ldots, n$ is defined as

$$Y_k^n = Y_{k-1}^n + f(Y_{k-1}^n)\Delta t_k + g(Y_{k-1}^n)\Delta B_k^H + \frac{1}{2}\, g(Y_{k-1}^n) g'(Y_{k-1}^n) \left(\Delta B_k^H\right)^2,$$

where g' denotes the derivative of g and $Y_0^n = X_0$.

The following result allows us to assert that if we replace the solution by its Milstein approximation the estimators of H remain consistent.

2.9 Theorem. *Let f be a Lipschitz-continuos function and let $g \in \mathcal{C}^{1+\alpha}$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. Define*

$$\widehat{H}_n^{(1),M} := \frac{1}{2} - \frac{1}{2\ln(N_{2n}/N_n)} \ln \frac{V_{N_{2n}}^{(1)}(Y^n, 2)}{V_{N_n}^{(1)}(Y^n, 2)},$$
$$\widehat{H}_n^{(2),M} := \frac{1}{2} - \frac{1}{2\ln(N_{2n}/N_n)} \ln \frac{V_{N_{2n}}^{(2)}(Y^n, 2)}{V_{N_n}^{(2)}(Y^n, 2)}.$$

If the subdivisions of the interval $[0, T]$ satisfy the conditions of Theorem 2.6, then $\widehat{H}_n^{(1),M} - H \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. If subdivisions of the interval $[0, T]$ satisfy the conditions of Theorem 2.8, then $\widehat{H}_n^{(2),M} - H \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.

Proof of Theorem 2.9. We consider only the second order increments of the Milstein approximation of the solution X of the equation (1.1). A consideration of the first order increments of the Milstein approximation is quite similar. Note that at subdivision points $\{t_k^n\}$ the Milstein approximation can be written in the form

$$\begin{aligned} Y^n(t_k^n) =& Y^n(t_{k-1}^n) + f(Y^n(t_{k-1}^n)) \cdot \Delta t_k^n + g(Y^n(t_{k-1}^n)) \cdot \Delta B_k^{H,n} \\ &+ \frac{1}{2} g(Y^n(t_{k-1}^n)) g'(Y^n(t_{k-1}^n)) \cdot \left(\Delta B_k^{H,n}\right)^2, \end{aligned}$$

$Y^n(0) = X(0) = \xi$. Denote $Y_k^n = Y^n(t_k^n)$ and $gg'(Y_k^n) = g(Y_k^n) g'(Y_k^n)$. Then

$$\begin{aligned} \Delta_{ir}{}^{(2)} Y_k^n &= \Delta t_k^n Y_{k+1}^n + \Delta t_{k+1}^n Y_{k-1}^n - (\Delta t_k^n + \Delta t_{k+1}^n) Y_k^n \\ &= \Delta t_k^n (Y_{k+1}^n - Y_k^n) - \Delta t_{k+1}^n (Y_k^n - Y_{k-1}^n) \\ &= \Delta t_k^n \left[f(Y_k^n) \Delta t_{k+1}^n + g(Y_k^n) \Delta B_{k+1}^{H,n} + \frac{1}{2} gg'(Y_k^n) (\Delta B_{k+1}^{H,n})^2 \right] \\ &\quad - \Delta t_{k+1}^n \left[f(Y_{k-1}^n) \Delta t_k^n + g(Y_{k-1}^n) \Delta B_k^{H,n} + \frac{1}{2} gg'(Y_{k-1}) (\Delta B_k^{H,n})^2 \right]. \end{aligned}$$

Therefore $\Delta_{ir}^{(2)} Y_k^n$ can be rewritten as

$$\begin{aligned} \Delta_{ir}{}^{(2)} Y_k^n = & \Delta t_k^n \Delta t_{k+1}^n [f(Y_k^n) - f(Y_{k-1}^n)] \\ & + [g(Y_k^n) \Delta t_k^n \Delta B_{k+1}^{H,n} - g(Y_{k-1}^n) \Delta t_{k+1}^n \Delta B_k^{H,n}] \\ & + \frac{1}{2} [gg'(Y_k^n) \Delta t_k^n (\Delta B_{k+1}^{H,n})^2 - gg'(Y_{k-1}^n) \Delta t_{k+1}^n (\Delta B_k^{H,n})^2] \end{aligned}$$

and, therefore

$$\begin{aligned}\Delta_{ir}{}^{(2)} Y_k^n &= \Big[\, [f(Y_k^n) - f(Y_{k-1}^n)]\, \Delta t_k^n \Delta t_{k+1}^n \\ &\qquad + [g(Y_k^n) - g(Y_{k-1}^n)]\, \Delta t_{k+1}^n \Delta B_k^{H,n} \Big] \\ &\quad + \frac{1}{2}[gg'(Y_k^n)\Delta t_k^n (\Delta B_{k+1}^{H,n})^2 - gg'(Y_{k-1}^n)\Delta t_{k+1}^n (\Delta B_k^{H,n})^2] \\ &\quad + g(Y_k^n)\Delta_{ir}^{(2)} B_k^{H,n} \\ &= I_{n,k}^{(1)} + \frac{1}{2} I_{n,k}^{(2)} + g(Y_k^n)\Delta_{ir}^{(2)} B_k^{H,n}\,.\end{aligned}$$

Further

$$\begin{aligned}&(I_{n,k}^{(1)})^2 \leqslant 2 \max_{1\leqslant k\leqslant N_n} |Y_k^n - Y_{k-1}^n|^2 \Big[m_n^4 L^2 + m_n^2 |g'|_\infty^2 \max_{1\leqslant k\leqslant N_n} |\Delta B_k^{H,n}|^2\Big], \\ &|I_{n,k}^{(1)}[g(Y_k^n)\Delta_{ir}^{(2)} B_k^{H,n}]| \leqslant \\ &\quad \leqslant 2m_n^2 \max_{1\leqslant k\leqslant N_n} |g(Y_k^n)| \max_{1\leqslant k\leqslant N_n} |Y_k^n - Y_{k-1}^n| \max_{1\leqslant k\leqslant N_n} |\Delta B_k^{H,n}| \\ &\qquad \times \Big[L m_n + |g'|_\infty \max_{1\leqslant k\leqslant N_n} |\Delta B_k^{H,n}|\Big], \\ &(I_{n,k}^{(2)})^2 \leqslant 4m_n^2 \max_{1\leqslant k\leqslant N_n} |gg'(Y_k^n)|^2 \max_{1\leqslant k\leqslant N_n} |\Delta B_k^{H,n}|^4\,.\end{aligned}$$

By using the inequality (1.8) we get

$$\begin{aligned}&\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_k^n (I_{n,k}^{(1)})^2 \leqslant \\ &\quad \leqslant 2T \max_{1\leqslant k\leqslant N_n} |Y_k^n - Y_{k-1}^n|^2 \Big[\frac{m_n^4 L^2}{p_n^{2+2H}} + \frac{m_n^{2+2/p}|g'|_\infty^2}{p_n^{2+2H}} (L_T^{H,1/p})^2\Big],\end{aligned}$$

$$\begin{aligned}&2\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_k^n |I_{n,k}^{(1)}[g(Y_k^n)\Delta_{ir}^{(2)} B_k^{H,n}]| \leqslant \\ &\quad \leqslant 4T \max_{1\leqslant k\leqslant N_n} |g(Y_k^n)| \max_{1\leqslant k\leqslant N_n} |Y_k^n - Y_{k-1}^n| L_T^{H,1/p} \\ &\qquad \times \Big[\frac{m_n^{3+1/p} L}{p_n^{2+2H}} + \frac{m_n^{2+2/p}|g'|_\infty}{p_n^{2+2H}} L_T^{H,1/p}\Big],\end{aligned}$$

$$\frac{1}{4}\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_k^n(I_{n,k}^{(2)})^2 \leqslant T\,\frac{m_n^{2+4/p}}{p_n^{2+2H}}\max_{1\leqslant k\leqslant N_n}|gg'(Y_k^n)|^2(L_T^{H,1/p})^4$$

and

$$\begin{aligned}&\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_k^n\big|[I_{n,k}^{(1)}+g(Y_k^n)\Delta_{ir}^{(2)}B_k^{H,n}]I_{n,k}^{(2)}\big|\\&\quad\leqslant\sqrt{2\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_k^n[|I_{n,k}^{(1)}|^2+g^2(Y_k^n)(\Delta_{ir}^{(2)}B_k^{H,n})^2]\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_k^n|I_{n,k}^{(2)}|^2}\,.\end{aligned}$$

It is easily verified that

$$\begin{aligned}\max_{1\leqslant k\leqslant N_n}|Y_k^n-Y_{k-1}^n| \leqslant &\max_{1\leqslant k\leqslant N_n}|f(Y_{k-1}^n)|\cdot\Delta t_k^n+\max_{1\leqslant k\leqslant N_n}|g(Y_{k-1}^n)\cdot\Delta B_k^{H,n}|\\&+\frac{1}{2}\max_{1\leqslant k\leqslant N_n}\Big|gg'(Y_{k-1}^n)(\Delta B_k^{H,n})^2\Big|\\\leqslant &m_n^{1/p}\Big[\max_{1\leqslant k\leqslant N_n}|f(Y_{k-1}^n)|+\max_{1\leqslant k\leqslant N_n}|g(Y_{k-1}^n)|\cdot L_T^{H,1/p}\Big]\\&+m_n^{2/p}\max_{1\leqslant k\leqslant N_n}|gg'(Y_{k-1}^n)|\cdot(L_T^{H,1/p})^2\,.\end{aligned}$$

Since $\sup_n V_p\big(Y^n;[0,T]\big)<\infty$ (see Kubilius (1999)), the functions $g(Y_k^n)$, $g'(Y_k^n)$ and $gg'(Y_k^n)$ are uniformly bounded. Thus

$$V_{\pi_n}^{(2)}(Y^n,2)-\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_{k+1}^n g^2(Y_k^n)\left(\Delta_{ir}^{(2)}B_k^{H,n}\right)^2\xrightarrow{\mathbf{a.s.}}0\qquad\text{as}\quad n\to\infty.$$

To complete the proof, it suffices to observe that

$$\begin{aligned}&\left|\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_{k+1}^n[g^2(X_k^n)-g^2(Y_k^n)]\left(\Delta_{ir}^{(2)}B_k^{H,n}\right)^2\right|\\&\quad\leqslant\max_{1\leqslant k\leqslant N_n}|g^2(X_k^n)-g^2(Y_k^n)|\sum_{k=1}^{N_n-1}\mu_k^n\Delta t_{k+1}^n\left(\Delta_{ir}^{(2)}B_k^{H,n}\right)^2\end{aligned}$$

and the last term tends to 0 as $n\to\infty$. To prove this we use some well known results. In Bégyn (2005) it was proved that

$$\sum_{k=1}^{N_n-1} \mu_k^n \Delta t_{k+1}^n \left(\Delta_{ir}^{(2)} B_k^{H,n}\right)^2 \xrightarrow{\text{a.s.}} 2\int_0^T h(\ell(s))\,ds \qquad \text{as} \quad n\to\infty.$$

The p-variation distance between the solution and its Milstein approximation was estimated in Kubilius (1999) (see also Kubilius (2000)). It follows from this that

$$\sup_{t\leqslant T} |Y_t^n - X_t^n| \xrightarrow{\text{a.s.}} 0 \qquad \text{as} \quad n\to\infty.$$

Consequently, the limit behavior of the second order increments of the Milstein approximation is the same as the limit behavior of the solution of SDE. Thus, the convergence $\widehat{H}_n^{(2),M} - H \overset{\text{a.s.}}{\to} 0$ holds as $n\to\infty$.

2.4. The increment ratios statistic

The increment ratios (IR) statistic is defined as

$$R^{p,n}(f) = \frac{1}{n-p}\sum_{k=0}^{n-p-1} \frac{|\Delta_k^{p,n} f + \Delta_{k+1}^{p,n} f|}{|\Delta_k^{p,n} f| + |\Delta_{k+1}^{p,n} f|},$$

where $\Delta_k^{p,n} f$ denotes the p-order increments of a real-valued function f at t_k^n, $p = 1, 2, \ldots, k = 0, 1, \ldots, n-p$, that is,

$$\Delta_k^{1,n} f = f(t_{k+1}^n) - f(t_k^n), \qquad \Delta_k^{p,n} f = \Delta_k^{1,n}\Delta_k^{p-1,n} f.$$

J. M. Bardet and D. Surgailis (2010) showed that if X is a fractional Brownian motion (B^H) with parameter $H \in (0, 1)$, then

$$R^{p,n}(f) \xrightarrow{\text{a.s.}} \Lambda_p(H) \quad \text{as } n\to\infty,\ p = 1, 2, \tag{2.4}$$

$$\text{where} \quad \Lambda_p(H) = \mathbf{E}\frac{|\Delta_0^p B^H + \Delta_1^p B^H|}{|\Delta_0^p B^H| + |\Delta_1^p B^H|}.$$

The $R^{2,n}(f)$ statistic is better suited for practical purposes than $R^{1,n}(f)$ since the error arising from approximating $\Lambda_2(H)$ with a line is considerably lower than that of $\Lambda_1(H)$. Fig. 2.1 presents the graph of $\Lambda_2(H)$ as well as the graph of $R^{2,100}(B^H)$ averaged over 50 sample paths.

In the recent years it has been proposed by several authors to replace the

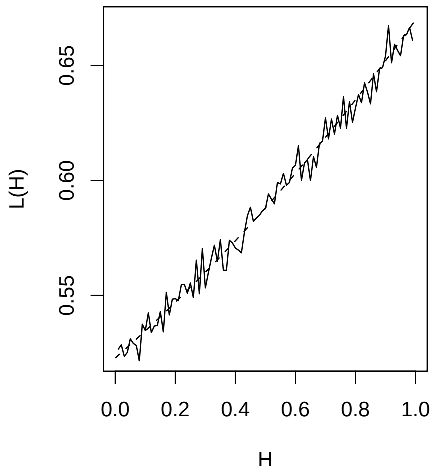

Fig. 2.1. Graphs of $\Lambda_2(H)$ and $R^{2,100}(B^H)$

classic Black–Scholes model, based on the standard geometric Brownian motion, with its fractional counterpart. This would enable the model to handle the possible existence of long-range dependance in the observed data. In this section it is shown that the convergence (2.4) holds for $H \in (1/2,\, 7/8)$ when $p = 1$ and $H \in (1/2,\, 1)$ when $p = 2$ if X is the solution of the fractional Black–Scholes equation.

2.10 Theorem. *Let* $B^H = \{B_t^H;\ t \in [0,\, 1]\}$ *denote the fractional Brownian motion with parameter* H *and let* X *be the solution of the fractional Black–Scholes equation (or, in other words, the fractional geometric Brownian motion)*

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t^H, \quad \mu, \sigma, X_0 \in \mathbb{R} \tag{2.5}$$

observed at times $t_k^n = \frac{k}{2^n}$*,* $k = 0,\, 1,\, \ldots,\, 2^n$*. Then*

$$R^{p,n}(X) \xrightarrow{\mathbf{a.s.}} \Lambda_p(H) \quad \text{as } n \to \infty,\ p = 1,\, 2$$

for $H \in (1/2,\, 7/8)$ *when* $p = 1$ *and* $H \in (1/2,\, 1)$ *when* $p = 2$*.*

The proof of this theorem is based on the following lemma which is a generalization of the corresponding lemma in the paper of J. M. Bardet and D. Surgailis.

2.1 Lemma. *Let* $\psi(x_1, x_2) = \frac{|x_1+x_2|}{|x_1|+|x_2|}$*,* $x_1, x_2 \in \mathbb{R}$*, and let* (Z_1, Z_2) *be a*

Gaussian vector with zero mean and dispersion $\mathbf{E}Z_i^2 = 1$, $i = 1, 2$. *Then for any r. v.* ξ_i, $i = 1, 2$,

$$\mathbf{E}\big|\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)\big| \leqslant 16 \max_{i=1,2} \sqrt[3]{\mathbf{E}\xi_i^2}, \quad k \geqslant 1.$$

Proof of Lemma 2.1. Let $\delta^2 = \max_{i=1,2} \mathbf{E}\xi_i^2$. Denote $U := \psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2) = U_\delta + U_\delta^c$,

$$\begin{aligned} U_\delta :=& U\mathbf{1}_{A_\delta} = \big(\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)\big)\mathbf{1}_{A_\delta}, \\ U_\delta^c :=& U\mathbf{1}_{A_\delta^c} = \big(\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)\big)\mathbf{1}_{A_\delta^c}, \end{aligned}$$

where $\mathbf{1}_{A_\delta}$ is the indicator of the event

$$A_\delta := \big\{|Z_1| > \delta^{2/3},\ |Z_2| > \delta^{2/3},\ |\xi_1| < \delta^{2/3}/2,\ |\xi_2| < \delta^{2/3}/2\big\}$$

and $\mathbf{1}_{A_\delta^c} = 1 - \mathbf{1}_{A_\delta}$ is the indicator of the complementary event A_δ^c. Clearly,

$$\begin{aligned} \mathbf{E}|U_\delta^c| \leqslant& 2\big[\mathbf{P}\big(|Z_1| < \delta^{2/3}\big) + \mathbf{P}\big(|Z_2| < \delta^{2/3}\big) \\ &+ \mathbf{P}\big(|\xi_1| \geqslant \delta^{2/3}/2\big) + \mathbf{P}\big(|\xi_2| \geqslant \delta^{2/3}/2\big)\big] \\ \leqslant& \frac{8}{\sqrt{2\pi}}\,\delta^{2/3} + 8 \max_{i=1,2} \frac{\mathbf{E}|\xi_i|^2}{\delta^{4/3}} \leqslant 12\delta^{2/3}. \end{aligned}$$

It remains to estimate $\mathbf{E}|U_\delta|$. By the mean value theorem,

$$U_\delta = \left(\xi_1 \frac{\partial\psi}{\partial x_1}(Z_1 + \theta\xi_1, Z_2 + \theta\xi_2) + \xi_2 \frac{\partial\psi}{\partial x_2}\,(Z_1 + \theta\xi_1, Z_2 + \theta\xi_2)\right)\mathbf{1}_{A_\delta},$$

where $0 < \theta(\omega) < 1$ and

$$\begin{aligned} \left|\frac{\partial\psi}{\partial x_i}(x_1, x_2)\right| &= \frac{|(|x_1| + |x_2|)\,\mathrm{sgn}(x_1 + x_2) - |x_1 + x_2|\,\mathrm{sgn}(x_i)|}{(|x_1| + |x_2|)^2} \\ &\leqslant \frac{2}{|x_1| + |x_2|}. \end{aligned}$$

Thus

$$\left|\frac{\partial\psi}{\partial x_i}(Z_1 + \theta\xi_1, Z_2 + \theta\xi_2)\right|\mathbf{1}_{A_\delta} \leqslant \frac{2}{|Z_1 + \theta\xi_1| + |Z_2 + \theta\xi_2|}\mathbf{1}_{A_\delta}$$

$$\leqslant \frac{2}{|Z_1+\theta\xi_1|+|Z_2+\theta\xi_2|}\mathbf{1}_{B_\delta} \leqslant \frac{2}{\delta^{2/3}}\mathbf{1}_{B_\delta},$$

where

$$B_\delta = \{|Z_1+\theta\xi_1| > \delta^{2/3}/2,\ |Z_1+\theta\xi_1| > \delta^{2/3}/2,\\ |\xi_1| \leqslant \delta^{2/3}/2,\ |\xi_2| \leqslant \delta^{2/3}/2\}.$$

Therefore

$$\mathbf{E}|U_\delta| \leqslant \mathbf{E}^{1/2}\xi_1^2\cdot\mathbf{E}^{1/2}\left[\left|\frac{\partial\psi}{\partial x_1}(Z_1+\theta\xi_1, Z_2+\theta\xi_2)\right|^2\mathbf{1}_{A_\delta}\right]\\ +\mathbf{E}^{1/2}\xi_2^2\cdot\mathbf{E}^{1/2}\left[\left|\frac{\partial\psi}{\partial x_2}(Z_1+\theta\xi_1, Z_2+\theta\xi_2)\right|^2\mathbf{1}_{A_\delta}\right] \leqslant 4\delta^{2/3}$$

and

$$\mathbf{E}|U| \leqslant 16\delta^{2/3}.$$

Proof of Theorem 2.10. Let Δt^n denote the mesh of the subdivision, that is, $\Delta t^n := 2^{-n}$. Let $\psi_k^n = \mu\Delta t^n + \sigma\Delta B_k^{H,n}$. The fractional geometric Brownian motion (2.5) is $X_t = c\exp(\mu t + \sigma B_t^H)$. Therefore $R^{1,n}(X)$ can be rewritten as

$$R^{1,n}(X) = \frac{1}{2^n-1}\sum_{k=0}^{2^n-2}\frac{|\exp(\psi_{k+1}^n)-\exp(-\psi_k^n)|}{|1-\exp(-\psi_k^n)|+|\exp(\psi_{k+1}^n)-1|}.$$

For briefness, let the index n be omitted. Then the Taylor expansion yields

$$\exp(\psi_{k+1})-1 = \sigma\Delta B_{k+1}^H + (\mu\Delta t + R(\mu\Delta t+\sigma\Delta B_{k+1}^H)),\\ 1-\exp(-\psi_k) = \sigma\Delta B_k^H + (\mu\Delta t - R(-\mu\Delta t-\sigma\Delta B_k^H)),$$

$$\exp(\psi_{k+1})-\exp(-\psi_k) = \sigma(\Delta B_{k+1}^H+\Delta B_k^H)+\\ +(2\mu\Delta t + R(\mu\Delta t+\sigma\Delta B_{k+1}^H) - R(-\mu\Delta t-\sigma\Delta B_k^H)),$$

where

$$R(x) = \frac{x^2}{2}e^{\theta x},\quad 0<\theta<1.$$

Let $Z_1(k) = \sigma\Delta B_k^H$, $\xi_1(k) = \mu\Delta t - R(-\mu\Delta t-\sigma\Delta B_k^H)$, $Z_2(k) = \sigma\Delta B_{k+1}^H$,

$\xi_2(k) = \mu\Delta t + R(\mu\Delta t + \sigma\Delta B^H_{k+1})$. Obviously

$$R^{1,n}(X) = \frac{1}{2^n-1}\sum_{k=0}^{2^n-2}\frac{|Z_1(k)+\xi_1(k)+Z_2(k)+\xi_2(k)|}{|Z_1(k)+\xi_1(k)|+|Z_2(k)+\xi_2(k)|}.$$

Define $\psi_k := \mu\Delta t + \sigma\Delta B^H_k$. Then

$$\begin{aligned}\mathbf{E}(\xi_1(k))^2 &\leqslant 2(\mu\Delta t)^2 + \mathbf{E}\big(\psi_k^4 e^{-2\theta\psi_k}\big)\\ &\leqslant 2(\mu\Delta t)^2 + \sqrt{105(\mathbf{E}\psi_k^2)^4\cdot\mathbf{E}e^{-4\theta\psi_k}}\\ &\leqslant 4\sqrt{105}\big[(\mu\Delta t)^4 + \sigma^4(\Delta t)^{4H}\big]\sqrt{\mathbf{E}e^{-4\theta\psi_k}}\end{aligned}$$

since ψ_k are Gaussian and therefore $\mathbf{E}\psi_k^8 = 105(\mathbf{E}\psi_k^2)^4$. Further,

$$\mathbf{E}e^{-4\theta\psi_k} \leqslant \mathbf{E}e^{4|\psi_k|} \leqslant e^{4|\mu|\Delta t}\mathbf{E}e^{4|\sigma\Delta B^H_k|} \leqslant 2e^{4|\mu|+8\sigma^2},$$

which yields that

$$\begin{aligned}\mathbf{E}(\xi_1(k))^2 &\leqslant 4\sqrt{210}\big[(\mu\Delta t)^4 + \sigma^4(\Delta t)^{4H}\big]e^{2|\mu|+4\sigma^2}\\ &= 4\sqrt{210}(\Delta t)^2\big[\mu^4(\Delta t)^2 + \sigma^4(\Delta t)^{2(2H-1)}\big]e^{2|\mu|+4\sigma^2}.\end{aligned}$$

Since $2H-1>0$, we get that $\mathbf{E}(\xi_1(k))^2 = \mathcal{O}(\Delta t)^2$. Similarly, $\mathbf{E}(\xi_2(k))^2 = \mathcal{O}(\Delta t)^2$ and according to Lemma 1

$$\begin{aligned}&\mathbf{E}\left|\frac{|Z_1(k)+\xi_1(k)+Z_2(k)+\xi_2(k)|}{|Z_1(k)+\xi_1(k)|+|Z_2(k)+\xi_2(k)|} - \frac{|Z_1(k)+Z_2(k)|}{|Z_1(k)|+|Z_2(k)|}\right| =\\ &= \mathbf{E}\left|\frac{|\Delta X_k+\Delta X_{k+1}|}{|\Delta X_k|+|\Delta X_{k+1}|} - \frac{|\Delta B^H_k+\Delta B^H_{k+1}|}{|\Delta B^H_k|+|\Delta B^H_{k+1}|}\right| = \mathcal{O}(2^{-n})^{2/3}.\end{aligned}$$

Then

$$\begin{aligned}&\mathbf{E}|R^{1,n}(X) - R^{1,n}(B^H)|\\ &\leqslant \frac{1}{n-1}\sum_{k=0}^{n-2}\mathbf{E}\left|\frac{|\Delta X_k+\Delta X_{k+1}|}{|\Delta X_k|+|\Delta X_{k+1}|} - \frac{|\Delta B^H_k+\Delta B^H_{k+1}|}{|B^H_k|+|\Delta B^H_{k+1}|}\right| = \mathcal{O}(2^{-n})^{2/3}\end{aligned}$$

and, consequently, $R^{1,n}(X) \xrightarrow{\mathbf{P}} R^{1,n}(B^H)$ as $n\to\infty$.

Let $\zeta_n := R^{1,n}(X) - R^{1,n}(B^H)$. Then the Chebyshev's inequality yields

$$\mathbf{P}\big(|\zeta_n| > 2^{-n/3}\big) \leqslant 2^{n/3}\mathbf{E}|\zeta_n| \leqslant 2^{-n/3}$$

and

$$\sum_{n=1}^{\infty} \mathbf{P}\big(|\zeta_n| > 2^{-n/3}\big) \leqslant \sum_{n=1}^{\infty} 2^{-n/3} < \infty.$$

According to the Borel–Cantelli lemma,

$$\mathbf{P}\Big(\limsup_{n\to\infty} \{|\zeta_n| > 2^{-n/3}\}\Big) = 0$$

which implies that $R^{1,n}(X) \overset{\text{a.s.}}{\to} R^{1,n}(B^H)$, $n \to \infty$.

The convergence $R^{p,n}(B^H) \overset{\text{a.s.}}{\to} \Lambda_1(H)$, $n \to \infty$ is established in Bardet, Surgailis (2010) and holds for $H \in (0, 7/8)$ when $p = 1$ and $H \in (0, 1)$ when $p = 2$. Clearly, provided $R^{1,n}(X) \overset{\text{a.s.}}{\to} R^{1,n}(B^H)$ and $R^{p,n}(B^H) \overset{\text{a.s.}}{\to} \Lambda_1(H)$, $n \to \infty$ it follows that $R^{1,n}(X) \overset{\text{a.s.}}{\to} \Lambda_1(H)$, $n \to \infty$ which completes the proof for the case $p = 1$. The proof for $p = 2$ follows analogously.

Table 2.1. Mean squared errors $\cdot 10^2$.

Nsp	n	100	500	1000
10	B^H	1.0997	0.6383	0.3592
	X	1.1000	0.5891	0.3916
50	B^H	0.5043	0.2077	0.1537
	X	0.5817	0.2300	0.1598
100	B^H	0.3421	0.1781	0.1246
	X	0.3398	0.1682	0.1257

Fig. 2.2 presents the graph of $\Lambda_2(H)$ together with the graph of $R^{2,100}(X)$ averaged over 50 sample paths, X being the fractional geometric Brownian motion. Table 2.1 shows the comparison of mean squared errors of $R^{2,n}(X) - \Lambda_2(H)$ and $R^{2,n}(B^H) - \Lambda_2(H)$ for the sample path lengths $n = 100, 200, 500$ and the numbers of sample paths $Nsp = 20, 50, 100$. The coefficients of the fractional geometric Brownian motion were chosen as $X_0 = 1$, $\mu = -0.3$, $\sigma = 0.5$.

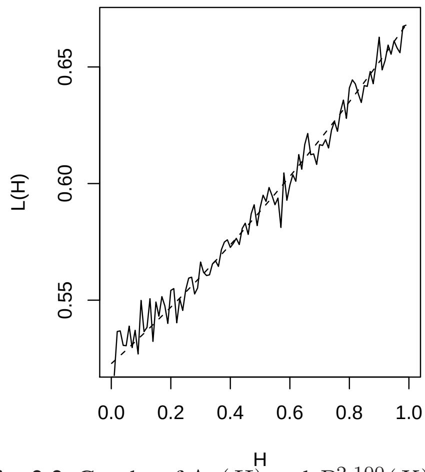

Fig. 2.2. Graphs of $\Lambda_2(H)$ and $R^{2,100}(X)$

2.5. The convergence rate of the Gladyshev estimator

In this section we define the modified Gladyshev's estimator of the fBm parameter H and derive the rate of convergence of it to its real value. To our knowledge, this problem is new and interesting from the practical point of view.

To recall, for a real-valued process $X = \{X_t;\ t \in [0,1]\}$ taking values at the points t_k^n, $k = 0, \ldots, N_n$, the first order quadratic variation is defined as

$$V_n^{(1)}(X,2) = \sum_{k=1}^{N_n} \left(\Delta X_k^n\right)^2, \quad \Delta X_k^n = X\left(t_k^n\right) - X\left(t_{k-1}^n\right).$$

Let B^H be the fractional Brownian motion with the Hurst index H. Set $t_k^n = k2^{-n}$, $k = 1, \ldots, 2^n$. It is known (see Gladyshev (1963)) that

$$2^{n(2H-1)} V_n^{(1)}\left(B^H, 2\right) \xrightarrow{\text{a.s.}} 1 \quad \text{as } n \to \infty.$$

This result yields that

$$\widetilde{H}_n = \frac{1}{2} - \frac{\ln V_n^{(1)}(B^H, 2)}{2n \ln 2}$$

is a strongly consistent estimator of H.

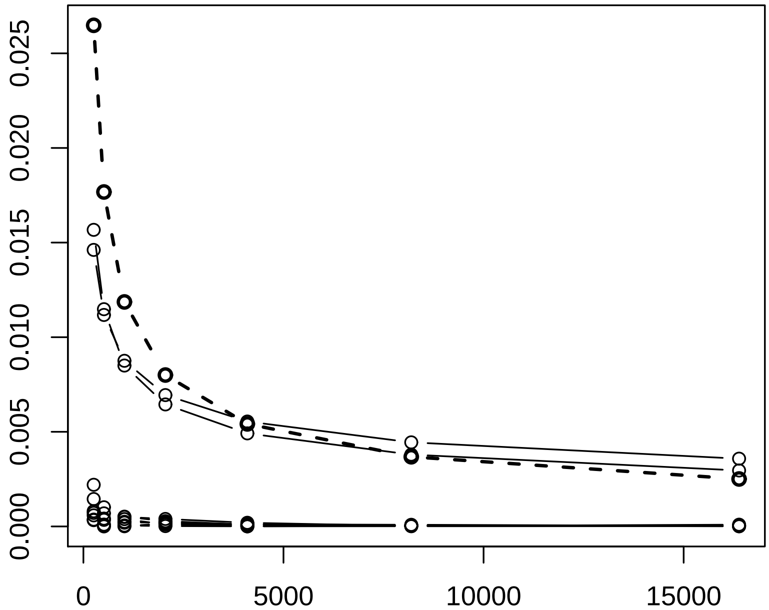

Fig. 2.3. Graphs of $\sqrt{N_n^{-1}\ln N_n}$ and $\mathbf{E}\big|\widehat{H}_n - H\big|$, $H \in \{0.55,\, 0.60,\, \ldots,\, 0.95\}$

Let us define a modified Gladyshev's estimator of the Hurst index H by

$$\widehat{H}_n = \left(\frac{1}{2} - \frac{\ln\left[V_n^{(1)}(B^H, 2)\right]}{2\ln N_n}\right)\mathbf{1}_{C_n},$$

for a regular subdivision π_n, where

$$C_n = \{V_n^{(1)}(B^H, 2) \geqslant N_n^{-2}\}.$$

The estimate $\widehat{H}_n$ is strongly consistent. Moreover, we can derive the rate of convergence of it to H. This follows from the next theorem. This result is illustrated by Fig. 2.3 which presents the graph of $\sqrt{N_n^{-1}\ln N_n}$ (the dotted line) and those of $\mathbf{E}|\widehat{H}_n - H|$ for $H \in (0.55,\, 0.60,\, \ldots,\, 0.95)$ against the sample path lengths $N_n \in \left\{2^k + 1;\, k = 8,\, \ldots,\, 14\right\}$. It can be seen that the convergence rate is determined by the behavior of $\widehat{H}_n$ for the values of H which are close to 1, as the lines closest to the dotted line correspond to $H = 0.9$ and $H = 0.95$.

2.11 Theorem. *Let B^H, $1/2 < H < 1$, be the fractional Brownian motion. $\widehat{H}_n$ is a strongly consistent estimator of the Hurst index H and the following* *rates of convergence hold*:

$$|\widehat{H}_n - H| = \mathcal{O}\big(\sqrt{N_n^{-1}\ln N_n}\,\big) \quad a.s. \;\; if \;\; \sum_{n=1}^{\infty} N_n^{-2} < \infty \tag{2.6}$$

and

$$\mathbf{E}|\widehat{H}_n - H| = \mathcal{O}\big(\sqrt{N_n^{-1}\ln N_n}\,\big)\,. \tag{2.7}$$

Proof of Theorem 2.11. First we have

$$\widehat{H}_n = H\mathbf{1}_{C_n} - \frac{\ln B_n}{2\ln N_n}\,\mathbf{1}_{C_n},$$

where $B_n = N_n^{2H-1}V_n^{(1)}(B^H, 2)$. Thus

$$\begin{aligned}|\widehat{H}_n - H| &\leqslant H\mathbf{1}_{\overline{C}_n} + \left|\frac{\ln B_n}{2\ln N_n}\right|\mathbf{1}_{\{B_n \geqslant N_n^{-2}\}}\\ &\leqslant H\mathbf{1}_{\{B_n < N_n^{-1}\}} - \frac{\ln B_n}{2\ln N_n}\,\mathbf{1}_{\{N_n^{-2}\leqslant B_n < 1\}} + \frac{\ln B_n}{2\ln N_n}\,\mathbf{1}_{\{B_n \geqslant 1\}}.\end{aligned}$$

Let (δ_n) be a sequence of positive numbers such that $\delta_n < 1$ and $\delta_n \downarrow 0$. The inequality $-\ln(1-x) \leqslant 20x$, $0 \leqslant x \leqslant 19/20$, yields

$$\begin{aligned}(-\ln B_n)\mathbf{1}_{\{1-\delta_n \leqslant B_n < 1\}} &= \big(-\ln\left[1-(1-B_n)\right]\big)\mathbf{1}_{\{1-\delta_n\leqslant B_n<1\}}\\ &\leqslant 20(1-B_n)\mathbf{1}_{\{1-\delta_n\leqslant B_n<1\}},\end{aligned}$$

if $\delta_n \leqslant 19/20$. So, it follows that

$$\begin{aligned}-\frac{\ln B_n}{2\ln N_n}\,\mathbf{1}_{\{N_n^{-2}\leqslant B_n<1\}} &\leqslant \mathbf{1}_{\{N_n^{-2}\leqslant B_n<1-\delta_n\}} + 10\,\frac{1-B_n}{\ln N_n}\,\mathbf{1}_{\{1-\delta_n\leqslant B_n<1\}}\\ &\leqslant \mathbf{1}_{\{N_n^{-2}\leqslant B_n<1-\delta_n\}} + \frac{10\delta_n}{\ln N_n}\,\mathbf{1}_{\{1-\delta_n\leqslant B_n<1\}}.\end{aligned} \tag{2.8}$$

The inequality $\ln(1+x) \leqslant x$, $x \geqslant 0$, yields

$$(\ln B_n)\mathbf{1}_{\{B_n\geqslant 1\}} = \big(\ln\left[1+(B_n-1)\right]\big)\mathbf{1}_{\{B_n\geqslant 1\}} \leqslant (B_n-1)\mathbf{1}_{\{B_n\geqslant 1\}}.$$

Thus

$$\frac{\ln B_n}{2\ln N_n}\,\mathbf{1}_{\{B_n\geqslant 1\}}\leqslant \frac{B_n-1}{2\ln N_n}\,\mathbf{1}_{\{1\leqslant B_n\leqslant 1+\delta_n\}}+\frac{B_n-1}{2\ln N_n}\,\mathbf{1}_{\{B_n>1+\delta_n\}}$$
$$\leqslant \frac{\delta_n}{2\ln N_n}\,\mathbf{1}_{\{1\leqslant B_n\leqslant 1+\delta_n\}}+\frac{B_n-1}{2\ln N_n}\,\mathbf{1}_{\{B_n>1+\delta_n\}}. \quad (2.9)$$

Inequalities (2.8) and (2.9) imply that

$$\big|\widehat{H}_n-H\big|\leqslant\Big(2+\frac{B_n-1}{2\ln N_n}\Big)\mathbf{1}_{\{|B_n-1|>\delta_n\}}+\frac{10\delta_n}{\ln N_n}. \quad (2.10)$$

To complete the proof it suffices to estimate the first term in the inequality (2.10) by using the Hanson and Wright inequality (Hanson, Wright (1971)). Note that $N_n^{2H-1}V_n^{(1)}\big(B^H,2\big)$ is the square of the Euclidean norm of an N_n-dimensional Gaussian vector X_n with components

$$N_n^{2H-1}\Delta B_k^{H,n}, \qquad 1\leqslant k\leqslant N_n.$$

Using a linear transformation of X_n one can get a new Gaussian vector Y_n with independent components. So there exist nonnegative real numbers $\{\lambda_{1,n},\,\ldots,\,\lambda_{N_n,N_n}\}$ and one N_n-dimensional Gaussian vector Y_n such that its components are independent standard Gaussian random variables and

$$N_n^{2H-1}V_n^{(1)}\big(B^H,2\big)=\sum_{j=1}^{N_n}\lambda_{j,N_n}\big(Y_n^{(j)}\big)^2.$$

Numbers $\{\lambda_{1,n},\,\ldots,\,\lambda_{N_n,N_n}\}$ are the eigenvalues of the symmetric $N_n\times N_n$-matrix

$$\Big(N_n^{2H-1}\mathbf{E}\big[\Delta B_j^{H,n}\Delta B_k^{H,n}\big]\Big)_{1\leqslant j,k\leqslant N_n}.$$

With the arguments of Gine, Klein (1975) and Bégyn (2005) one can get the inequality

$$\mathbf{P}\big(N_n^{2H-1}\big|V_n^{(1)}\big(B^H,2\big)-\mathbf{E}V_n^{(1)}\big(B^H,2\big)\big|\geqslant\varepsilon\big)\leqslant 2\exp\big(-K\varepsilon^2N_n\big), \quad (2.11)$$

which follows directly from the Hanson and Wright inequality, where $0<\varepsilon\leqslant 1$ and K is a positive constant. Set

$$\delta_n^2=\frac{2\ln N_n}{KN_n}\,.$$

From the inequality (2.11) it follows that

$$\mathbf{P}\big(|B_n - 1| > \delta_n\big) \leqslant \frac{2}{N_n^2}. \tag{2.12}$$

Obviously,

$$\mathbf{P}\bigg(\Big(2 + \frac{B_n - 1}{2\ln N_n}\Big)\mathbf{1}_{\{|B_n-1|>\delta_n\}} > 0\bigg) \leqslant \mathbf{P}\big(|B_n - 1| > \delta_n\big) \leqslant \frac{2}{N_n^2}.$$

Under the conditions of the theorem, the Borel–Cantelli lemma yields

$$\mathbf{P}\bigg(\limsup_{n\to\infty}\Big\{\Big(\frac{1}{2} + \frac{B_n - 1}{2\ln N_n}\Big)\mathbf{1}_{\{|B_n-1|>\delta_n\}} > 0\Big\}\bigg) = 0,$$

i. e.,

$$\Big(2 + \frac{B_n - 1}{2\ln N_n}\Big)\mathbf{1}_{\{|B_n-1|>\delta_n\}} = 0$$

for sufficiently large n. From the above results and the inequality (2.10) it follows that

$$\big|\widehat{H}_n - H\big| = \mathcal{O}\big(\sqrt{N_n^{-1}\ln N_n}\,\big) \quad \text{a.s.}$$

which completes the proof of (2.6). Note that from the inequalities (2.10) and (2.12) we get

$$\mathbf{E}\big|\widehat{H}_n - H\big| \leqslant \frac{2}{N_n^2} + \mathbf{E}\,\frac{|B_n - 1|}{2\ln N_n}\,\mathbf{1}_{\{|B_n-1|>\delta_n\}} + \frac{10\delta_n}{\ln N_n}.$$

It remaints to estimate the second term on the right side of the previous inequality. Note that

$$\begin{aligned}
&\mathbf{E}|B_n - 1|\,\mathbf{1}_{\{|B_n-1|>\delta_n\}} \\
&\quad \leqslant \mathbf{E}^{1/2}|B_n - 1|^2\sqrt{\mathbf{P}\big(|B_n - 1| > \delta_n\big)} \leqslant \frac{2}{N_n}\,\mathbf{E}^{1/2}(B_n^2 + 1) \\
&\qquad\qquad \leqslant \frac{2}{N_n}\bigg(N_n^{2H-1/2}\mathbf{E}^{1/2}\sum_{k=1}^{N_n}|\Delta B_k^{H,n}|^4 + 1\bigg) \\
&\qquad\qquad \leqslant \frac{2}{N_n}\big(\sqrt{3}\,N_n^{-1/2} + 1\big).
\end{aligned}$$

Thus

$$\mathbf{E}|\widehat{H}_n - H| \leqslant \frac{2}{N_n^2} + \frac{\sqrt{3}\,N_n^{-1/2} + 1}{N_n \ln N_n} + \frac{10\delta_n}{\ln N_n}\,,$$

which concludes the proof of (2.7).

2.6. Conclusions of the second chapter

1. Having proved the Theorem 1 and Theorem 3, the asymptotics of quadratic variations of the solution of the stochastic differential equation (1.1) in case of regularly spaced observations of the process were obtained. It was shown in Theorem 2 and Theorem 4 that the estimators of the Hurst index H originally obtained by Istas, Lang (1997) and Benassi *et al* (1998) for the fractional Brownian motion remain strongly consistent when the underlying process is the solution of the stochastic integral equation which is not necessarily Gaussian.
2. Having proved the Theorem 6 and Theorem 8, the asymptotics of quadratic variations of the solution of the stochastic differential equation (1.1) in case of irregularly spaced observations of the process were derived. It was shown in Theorem 7 that $\widetilde{H}^n_{dv1}$, the proposed estimator of the Hurst index H based on the first order quadratic variations is strongly consistent in case of irregularly spaced observations.
3. In Theorem 9 it was proved that the obtained estimators remain strongly consistent if the solution of the stochastic differential equation is replaced with its Milstein approximation.
4. In Theorem 10 it was proved that the increment ratio statistic can be applied to estimate the Hurst index H of the fractional geometric Brownian motion.
5. In Theorem 11 the rate of convergence of the modified Gladyshev estimator of the Hurst index H to its real value was derived.

3

Modelling of the estimators

The goal of this chapter is to compare the behavior of the estimators based on quadratic variations with some of the other known estimators, namely the naive and ordinary least squares Gladyshev and η-summing oscillation estimators, the variogram estimator and the IR estimator. These estimators are described in Section 3.2. Most of them were examined for Gaussian processes. The models chosen for comparison of these estimators were the fractional Ornstein-Uhlenbeck (O-U) process and the fractional geometric Brownian motion (gBm). The initial inference about the behavior of these estimators was drawn for the O-U process which is Gaussian, while the gBm process was used to check how the estimators behave in a non-Gaussian case.

In order to achieve that, a sufficient amount of sample paths of fBm is required. These sample paths were generated using the circulant matrix embedding method as described in Coeurjolly (2000) and the references therein. Let n denote the length of the sample path. The circulant matrix embedding method uses a fast Fourier transform which bypasses the matrix computations and therefore is sufficiently fast even for large values of n. 100 sample paths of the length $n = 2^{14} + 1$ were generated for each value of $H \in \{0.55, 0.6, \ldots, 0.95\}$ on the unit interval $t \in [0, 1]$.

The next step would be to use the generated fBm data to construct the sample paths of the considered processes. However, it's not always possible to find and use the explicit solution of the considered stochastic differential equation, therefore this solution might need to be replaced with its time discrete approximation. For a process X_t, its Milstein approximation at points t_k^n, $k = 1, \ldots, n$ is defined as

$$X_k^n = X_{k-1}^n + f(X_{k-1}^n)\Delta t_k + g(X_{k-1}^n)\Delta B_k^H + \frac{1}{2}\, g(X_{k-1}^n) g'(X_{k-1}^n) \left(\Delta B_k^H\right)^2,$$

where g' denotes the derivative of g. The fractional Ornstein-Uhlenbeck (O-U) and the fractional geometric Brownian motion (gBm) processes are defined as

$$dX_t = -\mu X_t\, dt + \sigma\, dB_t^H, \quad X_0 = c, \tag{O-U}$$

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t^H, \quad X_0 = c. \tag{gBm}$$

The solutions of these equations are, respectively,

$$X_t = e^{-\mu t}\left(c + \sigma \int_0^t e^{\mu s}\, dB_s^H\right) \quad \text{and} \quad X_t = c \exp\left(\mu t + \sigma B_t^H\right).$$

In fact, for the O-U process the Milstein approximation is reduced to the Euler one due to $g'(X_{k-1}^n) = (\sigma)' = 0$. The constants were chosen as $c = 1$, $\mu = 0.5$, $\sigma = 0.7$ in the O-U case and $c = 1$, $\mu = 0.2$, $\sigma = 0.5$ in the gBm case. The error introduced by using these approximated sample paths was negligible compared to the errors of the estimators themselves and will be ignored further on. All computations were performed using the R software environment (R Development Core Team (2009)).

3.1. Generation of the fractional Brownian motion

The algorithm to generate one sample path of the length n, using the circulant matrix embedding method, is as follows:

- Choose $M = 2^p \geqslant 2(n-1)$. Define the M-vector

$$V = \left(r(0), r(1), \ldots, r\left(\frac{M}{2} - 1\right), r\left(\frac{M}{2}\right), \right.$$
$$\left. , r\left(\frac{M}{2} - 1\right), \ldots, r(2), r(1)\right),$$

where

$$r(k) = \frac{1}{2n^{2H}} \left[|k+1|^{2H} - 2k^{2H} + |k-1|^{2H} \right]$$

is the autocovariance function of the fractional Gaussian noise.

- Compute $W = (w_1, \ldots, w_M)$, the fast Fourier transformation of V. All the coordinates of W must be non-negative. If this is not the case, the value of p must be increased until this requirement is met.
- Generate $U_j, V_j \sim \mathcal{N}(0,1)$ for all $1 \leqslant j < \frac{M}{2}$ and let $Z_1 = U_1$, $Z_{\frac{M}{2}+1} = V_1$,

  $$Z_j = \frac{1}{\sqrt{2}} (U_j + iV_j), \quad Z_{M+2-j} = \frac{1}{\sqrt{2}} (U_j - iV_j), \quad 1 < j \leqslant \frac{M}{2}.$$

  Then, define the M-vector U as

  $$U_k = \sqrt{w_k} Z_k, \quad k = 1, \ldots, M.$$

- Compute Y as an inverse fast Fourier transformation of the complex vector U and define X as

  $$X_k = X_{k-1} + \text{Re}(Y_k), \quad X_0 = 0, \quad k = 1, \ldots, n-1,$$

  $\text{Re}(Y)$ denoting the real part of the complex variable Y.

The obtained vector X is the desired sample path of the fractional Brownian motion with the Hurst index H.

3.2. Estimators

3.2.1. Discrete variation estimators

For a real-valued process $X = \{X_t;\, t \in [0,\, 1]\}$, we define the first and second order quadratic variations as

$$V_n^{(1)}(X,2) = \sum_{k=1}^{n} \left(\Delta_k^{(1)} X\right)^2, \qquad V_n^{(2)}(X,2) = \sum_{k=1}^{n-1} \left(\Delta_k^{(2)} X\right)^2,$$

where

$$\Delta_k^{(1)} X = X(t_k^n) - X(t_{k-1}^n)\,, \quad \Delta_k^{(2)} X = X(t_{k+1}^n) - 2X(t_k^n) + X(t_{k-1}^n)\,.$$

and $t_k^n = \frac{k}{n}$. Let X be the solution of (1.1). It is known (see Kubilius, Melichov (2008) - Kubilius, Melichov (2010)) that

$$\widehat{H}_{dv1}^n = \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}^{(1)}(X,2)}{V_n^{(1)}(X,2)}, \qquad \widehat{H}_{dv2}^n = \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}^{(2)}(X,2)}{V_n^{(2)}(X,2)}$$

are strongly consistent estimators of the Hurst index H, i.e.,

$$\widehat{H}_{dv1}^n - H \xrightarrow{\mathbf{a.s.}} 0 \quad \text{and} \quad \widehat{H}_{dv2}^n - H \xrightarrow{\mathbf{a.s.}} 0 \quad \text{as} \quad n \to \infty.$$

Here $V_{2n}^{(\cdot)}(X,2)$ corresponds to the quadratic variation of the whole sample path while $V_n^{(\cdot)}(X,2)$ is the variation of the subset $\{X_k : k = 2j,\, 0 \leqslant j \leqslant [n/2]\}$, $[x]$ denotes the integer part of x.

3.2.2. Gladyshev and η-summing oscillation estimators

The following estimators were described in R. Norvaiša and D.M. Salopek (2002, Norvaiša, Salopek (2002)). The ordinary least squares (OLS) Gladyshev and η-summing oscillation estimators require a sample path of the length $2^n + 1$, $n \in \mathbb{N}$, which dictated the length of our modeled sample paths. Define $\eta_M = \{N_m = 2^m\colon\ 1 \leqslant m \leqslant M\}$ and let

$$s(m) = \sum_{i=1}^{N_m} \left[X\left(\frac{i}{N_m}\right) - X\left(\frac{i-1}{N_m}\right)\right]^2.$$

The naive Gladyshev estimator of the Hurst index H is given by

$$\widehat{H}_{gn}^M = \frac{\log \sqrt{s(M) 2^{-M}}}{\log 2^{-M}}\,,$$

and the OLS Gladyshev estimator is given by

$$\widehat{H}_{go}^M = \frac{\sum_{m=1}^M (z_m - \bar{z})^2}{\sum_{m=1}^M (z_m - \bar{z})\, m}\,,$$

where $z_m = \log_2 \sqrt{2^m / s(m)}$ for $m \in \{1, \ldots, M\}$ and $\bar{z} = M^{-1} \sum_{m=1}^M z_m$.

For each $m \in \{1, \ldots, M\}$, define

$$Q(m) = \sum_{i=1}^{N_m} \left[\max_{t_k^n \in \Delta_{i,m}} \{X(t_k^n)\} - \min_{t_k^n \in \Delta_{i,m}} \{X(t_k^n)\} \right],$$

where

$$\Delta_{i,m} = \left[\frac{i-1}{N_m}, \frac{i}{N_m} \right].$$

The naive oscillation estimator is defined by

$$\widehat{H}_{osn}^M = \frac{\log_2(N_M/Q(M))}{\log_2 N_M}$$

and the OLS oscillation estimator is defined by

$$\widehat{H}_{oso}^M = \frac{\sum_{m=1}^M (z_m - \bar{z})^2}{\sum_{m=1}^M (z_m - \bar{z}) N_m}$$

where $z_m = \log_2 \sqrt{N_m/Q(m)}$ and $\bar{z} = M^{-1} \sum_{m=1}^M z_m$.

For $M = 14$ we simulate estimates defined above.

3.2.3. Variogram estimator

The variogram of the process $X = \{X_t, t \in [0, 1]\}$ for the lag ℓ is defined Chronopoulou, Viens (2010) as

$$V(\ell) = \mathbb{E}\left[(X_t - X_{t-\ell})^2 \right].$$

In order to estimate the Hurst index H, we choose a set of lags, in our case, it was $\{\ell = 2^i; i = 0, \ldots, 5\}$. Then $\widehat{H}_{var}^n = b/2$, where b is the slope of the linear regression line of $log(V(\ell))$ against $log(\ell)$.

3.2.4. Increment ratios estimator

This estimator was proposed by Bardet, Surgailis (2010). For the O-U or gBm process $X = \{X_t; t \in [0, 1]\}$ given at points $t_k^n = k/n$, $k = 0, 1, \ldots, n$, the increment ratios (IR) estimator of H can be computed using the approximated formula

$$\widehat{H}_{ir}^{n} = \frac{1}{0.1468}\left(\frac{1}{n-2}\sum_{k=1}^{n-2}\frac{\left|\Delta_k^{(2)}X + \Delta_{k+1}^{(2)}X\right|}{\left|\Delta_k^{(2)}X\right| + \left|\Delta_{k+1}^{(2)}X\right|} - 0.5174\right),$$

where $\Delta_k^{(2)}X = X(t_{k+1}^n) - 2X(t_k^n) + X(t_{k-1}^n)$.

3.3. The Ornstein-Uhlenbeck process

3.3.1. Dependance on the value of the Hurst index

The first goal of this section is to compare the behavior of these estimators for different values of the Hurst index H. Table 3.1 presents the biases $\overline{H} - H = \mathbf{E}(\widehat{H} - H)$ as well as the mean squared errors defined as $MSE(\widehat{H}) = \mathbf{E}(\widehat{H} - H)^2$ for the sample path lengths of, respectively, $2^{14}+1$ and $2^{10}+1$ points. Figure B.1 (see Appendix) illustrates this further presenting the boxplots of the considered estimators for the length of sample paths $n = 2^{14}+1$ points. Here and further in this section the figures related to the estimators $\widehat{H}_{gn}$ and $\widehat{H}_{go}$ are omitted, since their behavior does not significantly differ from the behavior of $\widehat{H}_{osn}$ and $\widehat{H}_{oso}$. The numbers printed in bold correspond to the estimators that performed better than the others for the specific value of H and the considered numeric characteristic.

It can be seen that the estimators $\widehat{H}_{dv1}$, $\widehat{H}_{var}$, $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$ exhibit increases of the biases and the mean squared errors for larger values of H. $\widehat{H}_{go}$ and $\widehat{H}_{oso}$ seem to be less dependant on that, however, they tend to slightly undervalue the Hurst index when it is close to 1. $\widehat{H}_{ir}$ tends to slightly undervalue H when $H < 3/4$ and to overvalue it when $H > 3/4$; the most likely cause of this is the numeric constants in the formula used for this estimator. The behavior of $\widehat{H}_{dv2}$ does not change noticeably for different values of H.

Another interesting observation is the, comparatively, very low mean squared errors of $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$ which they display as long as the Hurst index is not too close to 1. However these estimators also possess the largest bias. $\widehat{H}_{go}$ and $\widehat{H}_{oso}$, the OLS versions of these two estimators behave in a completely different way – they have smaller biases which are comparable to those of the other considered estimators, but this comes at the cost of heavily increased MSE.

Table 3.1. Comparison of the estimators for the O-U process.

H		0.55	0.7	0.8	0.95
MSE	dv1	0.008	**0.005**	**0.008**	**0.020**
	dv2	0.015	0.011	0.011	**0.010**
	var	**0.007**	0.009	0.014	0.024
	gn	**0.002**	**0.002**	**0.003**	**0.021**
	osn	**0.004**	**0.004**	**0.005**	0.025
	go	0.038	0.037	0.048	0.050
	oso	0.041	0.037	0.050	0.053
	ir	0.022	0.019	0.019	0.022
$\overline{H} - H$	dv1	**0.000**	**0.001**	**-0.001**	**-0.009**
	dv2	**0.000**	**0.001**	-0.002	**-0.001**
	var	**0.000**	**-0.001**	**0.000**	**-0.012**
	gn	0.037	0.037	0.037	0.041
	osn	0.060	0.060	0.060	0.063
	go	-0.013	-0.024	-0.019	-0.032
	oso	-0.004	-0.017	-0.009	-0.020
	ir	-0.019	-0.009	**-0.001**	0.028

(a) $N = 2^{14} + 1$

H		0.55	0.7	0.8	0.95
MSE	dv1	0.027	0.024	**0.026**	**0.030**
	dv2	0.050	0.054	0.050	0.044
	var	0.029	0.031	0.038	**0.040**
	gn	**0.006**	**0.006**	**0.010**	**0.040**
	osn	**0.010**	**0.011**	**0.015**	0.048
	go	0.092	0.061	0.071	0.068
	oso	0.093	0.060	0.073	0.071
	ir	0.074	0.067	0.069	0.077
$\overline{H} - H$	dv1	**0.000**	**-0.004**	**-0.001**	**-0.016**
	dv2	**0.001**	**0.000**	**-0.002**	**0.008**
	var	**0.000**	-0.012	-0.007	-0.025
	gn	0.051	0.052	0.052	0.062
	osn	0.084	0.084	0.084	0.091
	go	-0.010	-0.035	-0.030	-0.044
	oso	0.003	-0.023	-0.014	-0.028
	ir	-0.009	**-0.004**	**-0.004**	0.039

(b) $N = 2^{10} + 1$

3.3.2. Dependance on the length of the sample path

The second goal of this section is to compare the behavior of these estimators for different lengths of sample paths as well as to illustrate how the estimators' variances fluctuate as the length of the sample paths is increased. Table 3.2 shows the mean squared errors and the biases for the Hurst index values of 0.65 and 0.85, respectively. Figure B.2 (see Appendix) presents the boxplots of the estimators for $H = 0.85$.

The first obvious observation is that the bias of $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$ increases as the length of the sample paths is decreased. $\widehat{H}_{go}$ and $\widehat{H}_{oso}$ do not share this property, however their mean squared errors display only minor decreases when longer sample paths are taken. The other estimators show a rather regular decrease of their mean squared errors which is further illustrated by Figure B.3 (see Appendix) presenting the plots of $log(SD)$ against $log(n)$ for $H \in \{0.55, 0.6, \dots, 0.95\}$ where SD denotes the standard deviations.

Figure B.3 shows the rate at which the standard deviation decreases as the sample path length is increased. It can be seen that this rate depends on the value of H for all the estimators except $\widehat{H}_{dv2}$ and $\widehat{H}_{ir}$. The general trend is that this rate is lower for higher values of H which is most notable for $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$. On the other hand $\widehat{H}_{dv2}$ and $\widehat{H}_{ir}$ display no dependance of this kind.

Table 3.2. Comparison of the estimators for the O-U process for sample path lengths $N = 2^k + 1$.

	k	8	10	12	14
MSE	dv1	**0.053**	**0.025**	**0.014**	**0.007**
	dv2	0.102	0.055	0.027	0.013
	var	0.067	0.033	0.018	0.009
	gn	**0.013**	**0.006**	**0.003**	**0.002**
	osn	**0.021**	**0.011**	**0.006**	**0.004**
	go	0.084	0.063	0.049	0.039
	oso	0.086	0.064	0.050	0.040
	ir	0.172	0.079	0.037	0.018
$\overline{H} - H$	dv1	**-0.005**	**0.000**	**0.002**	**0.001**
	dv2	**-0.016**	**0.000**	**0.001**	**0.000**
	var	-0.021	**-0.003**	**-0.001**	**0.001**
	gn	0.064	0.051	0.043	0.037
	osn	0.105	0.084	0.070	0.060
	go	-0.036	-0.028	-0.021	-0.016
	oso	**-0.017**	-0.014	-0.011	-0.009
	ir	-0.043	-0.013	-0.014	-0.013

(a) $H = 0.65$

k	8	10	12	14
dv1	**0.045**	**0.030**	**0.019**	0.013
dv2	0.089	0.051	0.025	**0.012**
var	0.068	0.042	0.029	0.019
gn	**0.028**	**0.015**	**0.009**	**0.005**
osn	**0.036**	**0.020**	**0.012**	**0.008**
go	0.095	0.077	0.063	0.053
oso	0.102	0.082	0.067	0.057
ir	0.170	0.095	0.038	0.021
dv1	**-0.016**	**-0.008**	**-0.005**	**-0.003**
dv2	**-0.015**	**0.005**	**-0.002**	**-0.001**
var	-0.038	-0.018	**-0.010**	**-0.005**
gn	0.070	0.054	0.044	0.037
osn	0.110	0.086	0.071	0.061
go	-0.054	-0.043	-0.035	-0.028
oso	-0.033	-0.027	-0.022	-0.018
ir	-0.039	0.019	0.011	0.008

(b) $H = 0.85$

Also, if we consider the linear regression $log(SD) \sim log(n)$ for these two estimators, its slope is -0.5003 for $\widehat{H}_{dv2}$ and -0.5013 for $\widehat{H}_{ir}$, which suggests that for both these estimators $SD\left(\widehat{H}_{(\cdot)}\right) \sim \mathcal{O}(n^{-1/2})$.

3.4. The geometric Brownian motion

3.4.1. Dependance on the value of the Hurst index

Table 3.3 presents the mean squared errors and the biases for the sample path lengths of $2^{14}+1$ and $2^{10}+1$. Boxplots of these estimators for the sample path length $n = 2^{14} + 1$ can be found in Figure B.4 (see Appendix). It can be seen that, for the non-Gaussian gBm process, the estimators $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$ display higher biases for all the values of H.

An interesting observation is that, in the case of the O-U process the mean squared errors of $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$ were the lowest of all the considered estimators, while for the gBm model their mean squared errors surpassed those of $\widehat{H}_{dv1}$, $\widehat{H}_{dv2}$ and $\widehat{H}_{var}$.

Table 3.3. Comparison of the estimators for the gBm process.

H		0.55	0.7	0.8	0.95
MSE	dv1	**0.008**	**0.006**	**0.009**	**0.021**
	dv2	**0.016**	**0.012**	**0.011**	**0.010**
	var	**0.008**	**0.010**	0.015	0.025
	gn	0.032	0.030	0.034	0.049
	osn	0.035	0.032	0.037	0.054
	go	0.033	0.046	0.053	0.049
	oso	0.035	0.049	0.056	0.053
	ir	0.022	0.019	0.019	**0.022**
$\overline{H} - H$	dv1	**0.000**	**0.001**	**0.000**	**-0.006**
	dv2	**0.000**	**0.001**	**-0.002**	**0.000**
	var	**0.000**	**0.001**	**0.000**	**-0.009**
	gn	0.055	0.051	0.061	0.059
	osn	0.080	0.076	0.086	0.082
	go	-0.014	-0.016	-0.021	-0.026
	oso	**-0.003**	-0.006	-0.011	-0.014
	ir	-0.019	-0.009	**-0.001**	0.028

(a) $N = 2^{14} + 1$

H		0.55	0.7	0.8	0.95
MSE	dv1	**0.030**	**0.026**	**0.029**	**0.030**
	dv2	0.054	0.054	0.055	**0.046**
	var	**0.030**	**0.037**	**0.040**	**0.041**
	gn	0.047	0.044	**0.051**	0.079
	osn	0.053	0.048	0.058	0.088
	go	0.055	0.072	0.080	0.066
	oso	0.057	0.077	0.084	0.070
	ir	0.074	0.067	0.069	0.077
$\overline{H} - H$	dv1	**-0.001**	**-0.001**	**0.000**	**-0.012**
	dv2	**0.001**	**0.000**	**-0.002**	**0.007**
	var	**0.000**	-0.006	-0.008	-0.020
	gn	0.077	0.072	0.086	0.086
	osn	0.112	0.106	0.120	0.116
	go	-0.020	-0.027	-0.036	-0.038
	oso	**-0.001**	-0.009	-0.019	-0.020
	ir	-0.009	-0.005	**-0.004**	0.039

(b) $N = 2^{10} + 1$

The behavior of $\widehat{H}_{dv2}$ and $\widehat{H}_{ir}$ does not display notable differences for these two processes.

3.4.2. Dependance on the length of the sample path

Table 3.4 presents the mean squared errors and the biases for $H = 0.65$ and $H = 0.85$. Figure B.5 shows the boxplots of the estimators considered for $H = 0.85$, while Figure B.6 presents the plots of $log(SD)$ against $log(n)$ (see Appendix).

Compared to the O-U case the biases of $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$ are higher for all sample path lengths. The mean squared errors of $\widehat{H}_{gn}$, $\widehat{H}_{osn}$, $\widehat{H}_{go}$ and $\widehat{H}_{oso}$ are higher for all sample path lengths. In the case of relatively short sample paths ($2^8 - 2^{10}$) and $H > 3/4$, the estimators $\widehat{H}_{go}$ and $\widehat{H}_{oso}$ have at times severely overestimated the Hurst index H with the estimated value being higher than 2. Those values were excluded from their boxplots. The slope of the linear regression $log(SD) \sim log(n)$ is -0.5015 for $\widehat{H}_{dv2}$ and -0.5011 for $\widehat{H}_{ir}$, which does not differ significantly from the O-U case.

Table 3.4. Comparison of the estimators for the gBm process for sample path lengths $N = 2^k + 1$.

	k	8	10	12	14
MSE	dv1	**0.052**	**0.028**	**0.016**	**0.007**
	dv2	0.103	0.065	**0.028**	**0.013**
	var	0.073	**0.036**	**0.020**	**0.011**
	gn	**0.066**	0.050	0.041	0.034
	osn	0.075	0.056	0.044	0.037
	go	2.411	0.097	0.061	0.048
	oso	0.556	0.090	0.063	0.050
	ir	0.172	0.079	0.037	0.018
$\overline{H} - H$	dv1	**-0.002**	**0.001**	**0.002**	**0.001**
	dv2	-0.020	**-0.002**	**0.000**	**0.000**
	var	-0.013	**0.001**	**0.001**	**0.002**
	gn	0.093	0.074	0.062	0.054
	osn	0.137	0.110	0.092	0.079
	go	0.223	-0.009	-0.011	-0.010
	oso	0.065	0.006	**0.001**	**0.000**
	ir	-0.043	-0.013	-0.014	-0.013

(a) $H = 0.65$

	k	8	10	12	14
MSE	dv1	**0.047**	**0.033**	**0.023**	**0.016**
	dv2	0.094	**0.053**	**0.026**	**0.013**
	var	0.082	**0.050**	**0.032**	0.023
	gn	0.082	0.059	0.046	0.037
	osn	0.092	0.065	0.049	0.040
	go	0.104	0.085	0.071	0.060
	oso	0.109	0.090	0.075	0.064
	ir	0.169	0.095	0.038	0.021
$\overline{H} - H$	dv1	**-0.005**	**-0.001**	**0.001**	**0.000**
	dv2	-0.016	**0.003**	**-0.002**	**-0.001**
	var	-0.025	-0.007	**-0.002**	**0.001**
	gn	0.094	0.075	0.062	0.054
	osn	0.136	0.108	0.091	0.078
	go	-0.038	-0.028	-0.022	-0.017
	oso	-0.011	-0.009	-0.006	-0.004
	ir	-0.039	0.019	0.011	0.008

(b) $H = 0.85$

3.5. Conclusions of the modelling

1. The estimators $\widehat{H}_{gn}$ and $\widehat{H}_{osn}$, despite showing the least mean squared errors in the O-U case, have also shown much higher biases than other estimators considered in this section. This bias increases as the sample path length is decreased but shows no dependance on the value of the Hurst index H as long as H is not too close to 1. When $H > 0.9$, this bias increases further. In the gBm case the mean squared errors of these two estimators were greater than those of $\widehat{H}_{dv1}$, $\widehat{H}_{dv2}$ and $\widehat{H}_{var}$.
2. The estimators $\widehat{H}_{go}$ and $\widehat{H}_{oso}$, the ordinary least squares versions of the previous estimators, display totally different behavior - their biases are comparable with those of the other estimators. However, their mean squared errors are considerably higher than those of other estimators and tend to decrease only slightly as the sample path length is increased. Additionally, both of these estimators require the sample path length to be equal to $2^k + 1$, $k \in \mathbb{N}$, which means that, for sample paths of different length, some of the observations must be truncated.
3. The estimators $\widehat{H}_{dv1}$ and $\widehat{H}_{var}$ behaved differently for "small" and "large" values of H. As $H \in (1/2, 3/4)$, they displayed the best char-

acteristics while for higher values of H their performance was close to or worse than that of other estimators. $\widehat{H}_{var}$ displayed increased biases for shorter sample paths.

4. The characteristics of $\widehat{H}_{dv2}$ were slightly worse than that of $\widehat{H}_{dv1}$ and $\widehat{H}_{var}$ for shorter sample paths and $H < 3/4$, and they were similar or better for longer sample paths and $H > 3/4$. Also, it showed no notable dependance of its behavior on the value of H. $\widehat{H}_{ir}$ displayed such a dependance only for rather long sample paths, but its biases and mean squared errors were higher. Having considered the linear regression $log(SD) \sim log(n)$ for these two estimators, the results suggest that for both these estimators $SD\left(\widehat{H}_{(\cdot)}\right) \sim \mathcal{O}(n^{-1/2})$.

5. Calculation times for the estimators $\widehat{H}_{dv1}$, $\widehat{H}_{dv2}$ and $\widehat{H}_{osn}$ were about $0.02s$ with 100 sample paths of the length $N = 2^8 + 1$ and about $0.4s$ with 100 sample paths of the length $N = 2^{14} + 1$. Calculation times of $\widehat{H}_{gn}$ were about twice lower and those of $\widehat{H}_{go}$, $\widehat{H}_{oso}$ and $\widehat{H}_{ir}$ were 2–5 times higher.

General conclusions

Having solved the tasks listed in the introduction the following resultes were obtained:

1. Having proved the Theorems 1, 3, 6 and 8, the asymptotics of quadratic variations of the solution of the stochastic differential equation (1.1) were derived both in case of regularly and irregularly spaced observations.
2. In case of regularly spaced observations, in Theorem 2 and Theorem 4 it was proved that $\widehat{H}^n_{dv1}$ and $\widehat{H}^n_{dv2}$, the estimators of the Hurst index H originally obtained by Istas, Lang (1997) and Benassi *et al* (1998) for the fractional Brownian motion remain strongly consistent when the underlying process is the solution of the stochastic differential equation. In case of irregularly spaced observations, in Theorem 7 it was shown that $\widetilde{H}^n_{dv1}$, the proposed estimator of the Hurst index H based on the first order quadratic variations is strongly consistent.
3. In Theorem 9 it was proved that the obtained estimators remain strongly consistent if the solution of the stochastic differential equation is replaced with its Milstein approximation.

4. In Theorem 10 it was proved that the increment ratio statistic can be applied to estimate the Hurst index H of the fractional geometric Brownian motion.
5. In Theorem 11 the rate of convergence of the modified Gladyshev estimator of the Hurst index H to its real value was derived.
6. The obtained estimators were compared to some of the other known estimators, namely the naive and ordinary least squares Gladyshev and η-summing oscillation estimators, the variogram estimator and the IR estimator. The results of the modelling study suggest that if the value of the Hurst index is large ($H > 3/4$) or when the Hurst index is estimated from a sufficiently long sample path ($N > 2^{10}$), the $\widehat{H}_{dv2}$ estimator performs best. If either of these assumptions is not present, then $\widehat{H}_{dv1}$ and $\widehat{H}_{var}$ would likely provide a more precise estimate.

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Appendices

Appendix A. [R] source codes

Appendix A.1. Generation of the fractional Brownian motion

The function $genFBM(H, N)$ generates a single sample path of the fBm with the Hurst index H of the length N data points over the unit interval.

```
genFBM <- function(H,N) {
# M - length of the required vector of autocovariances
M <- 2^(ceil( 1 + log(N-1)/log(2) ))

# V is the M-vector of autocovariances of fGn
H2 <- 2*H
V <- c(0:(M/2), (M/2-1):1)
V <- 1/(2*N^(H2))*(abs(V-1)^(H2) - 2*V^(H2) + (V+1)^(H2))

# W is the fast Fourier transform of V
W <- Re(fft(V))

# We increase M until all coordinates of W are positive
while (any(W<=0) & (M<2^30)) {
M <- 2*M
V <- c(0:(M/2), (M/2-1):1)
V <- 1/(2*N^(H2))*(abs(V-1)^(H2) - 2*V^(H2) + (V+1)^(H2))
W <- Re(fft(V))
```

```
}

# X and Y and iid Gaussian with mean 0 and variance 1/sqrt(2)
X <- rnorm(M, mean=0, sd=(1/sqrt(2)))
Y <- rnorm(M, mean=0, sd=(1/sqrt(2)))
Z <- vector(length=M)
Z[1] <- X[1]
Z[M/2+1] <- Y[1]
Z[c(2:(M/2))] <- X[c(2:(M/2))] + 1i*Y[c(2:(M/2))]
Z[M+2-c(2:(M/2))] <- X[c(2:(M/2))] - 1i*Y[c(2:(M/2))]
U <- sqrt(W)*Z

# X is the fast Fourier transform of U = sqrt(W) * ( X + iY )
X <- fft(U, inverse=T)

# The real part of the first N coordinates of X is the desired
# sample path, in this case, the fGn.
# We calculate the fBm sample path as cumulated sums of fGn.
BHinc <- c(0,Re(X[1:N]))/sqrt(M)
BH <- cumsum(BHinc)
return(BH[-length(BH)])
}
```

Appendix A.2. Calculation of the estimators

The following functions estimate the value of the Hurst index H of a single sample path.

fH1 calculates the $\widehat{H}^n_{dv1}$ estimate:

```
fH1 <- function(X) {
X2n <- X
dX2n <- X2n[-1] - X2n[-length(X2n)]
Xn <- X2n[-c(seq(2,length(X2n),2))]
dXn <- Xn[-1] - Xn[-length(Xn)]
V2n <- sum(dX2n^2)
Vn <- sum(dXn^2)
H <- 1/2 - 1/(2*log(2)) * log(V2n/Vn)
return(H)
}
```

fH2 calculates the $\widehat{H}^n_{dv2}$ estimate:

```
fH2 <- function(X) {
X2n <- X
dX2n <- X2n[-1] - X2n[-length(X2n)]
d2X2n <- dX2n[-1] - dX2n[-length(dX2n)]
Xn <- X2n[-c(seq(2,length(X2n),2))]
dXn <- Xn[-1] - Xn[-length(Xn)]
```

```
d2Xn <- dXn[-1] - dXn[-length(dXn)]
V2n <- sum(d2X2n^2)
Vn <- sum(d2Xn^2)
H <- 1/2 - 1/(2*log(2)) * log(V2n/Vn)
return(H)
}
```

IR calculates the $\widehat{H}_{ir}^{n}$ estimate:

```
IR <- function(X) {
H <- (mean(abs(diff(diff(X[-1])) + diff(diff(X[-length(X)])))/
(abs(diff(diff(X[-1]))) + abs(diff(diff(X[-length(X)])))))
- 0.5174)/0.1468
}
```

G_naive calculates the $\widehat{H}_{gn}^{n}$ estimate:

```
G_naive <- function(X) {
m <- floor(log2(length(X)))
dX <- (X[-1] - X[-length(X)])^2
s2 <- sum(dX)
Hest <- log(sqrt(s2*2^(-m))) / log(2^(-m))
return(Hest)
}
```

G_OLS calculates the $\widehat{H}_{go}^{n}$ estimate:

```
G_OLS <- function(X) {
m <- floor(log2(length(X)))
mseq <- c(1:m)
s2 <- vector(length=m)
xm <- vector(length=m)
for (m in mseq) {
Xseq <- X[seq(1,length(X),(2^(floor(log2(length(X)))-m)))]
s2[m] <- sum((Xseq[-1] - Xseq[-length(Xseq)])^2)
xm[m] <- log2(sqrt((2^m)/(s2[m])))
}
xavg <- mean(xm)
Hest <- sum((xm-xavg)^2) / sum((xm-xavg)*mseq)
return(Hest)
}
```

OS_naive calculates the $\widehat{H}_{osn}^{n}$ estimate:

```
OS_naive <- function(X) {
M <- floor(log2(length(X)))
Q <- sum(pmax(X[-1],X[-length(X)]) - pmin(X[-1],X[-length(X)]))
Hest <- log2(2^M/Q) / log2(2^M)
return(Hest)
}
```

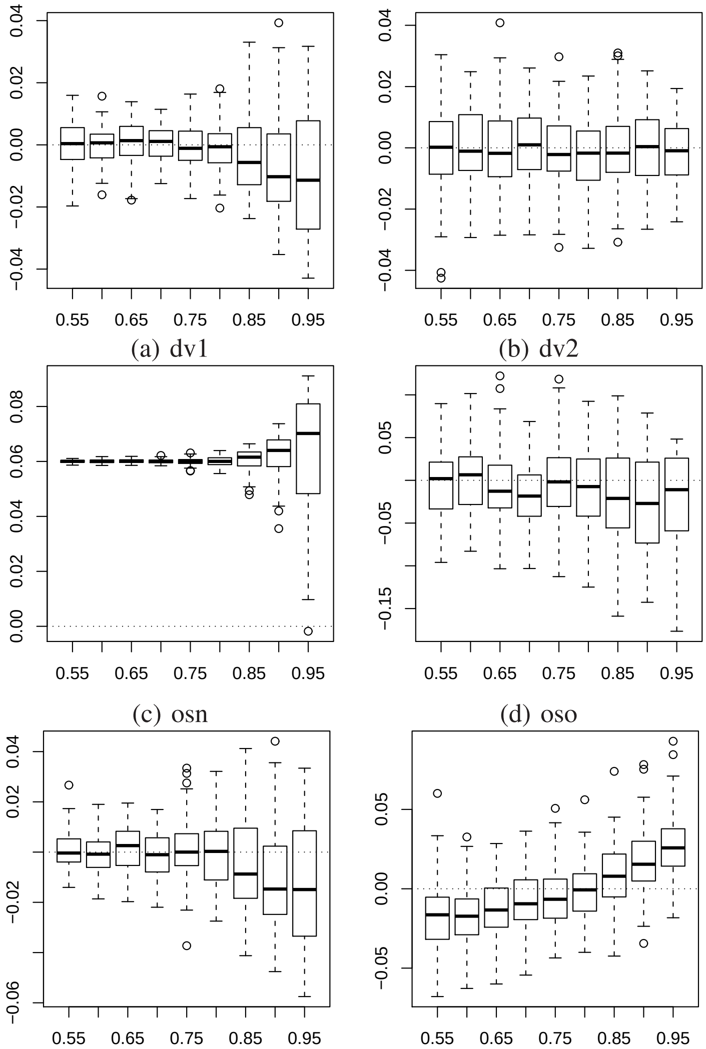
OS_OLS calculates the $\widehat{H}^n_{oso}$ estimate:

```
OS_OLS <- function(X) {
M <- floor(log2(length(X)))
nu <- 2^c(1:M)
xm <- vector(length=M)
for (k in c(1:M)) {
bsize <- (length(X)-1)/nu[k]
filt <- c(rep(c(TRUE,rep(FALSE,(bsize-1))),nu[k]),TRUE)
Y <- X[filt]
Q <- sum(pmax(Y[-1],Y[-length(Y)]) - pmin(Y[-1],Y[-length(Y)]))
xm[k] <- log2(nu[k]/Q)
}
xavg <- mean(xm)
Hest <- sum((xm-xavg)^2) / sum((xm-xavg)*log2(nu))
return(Hest)
}
```

Hvar calculates the $\widehat{H}^n_{var}$ estimate:

```
Hvar <- function(X) {
lag <- 2^c(0:5)
V <- vector(length=length(lag))
for (n in lag) {
dX <- X[-c(1:n)]-X[-c((length(X)-n+1):length(X))]
V[log2(n)+1] <- mean(dX^2)
}
H <- lm(log(V)~log(lag))$coefficients[2]/2
names(H)<-NULL
return(H)
}
```

Appendix B. Figures



(a) dv1 (b) dv2

(c) osn (d) oso

(e) var (f) ir

Fig. B.1. Boxplots for the O-U process, sample path length $n = 2^{14} + 1$.

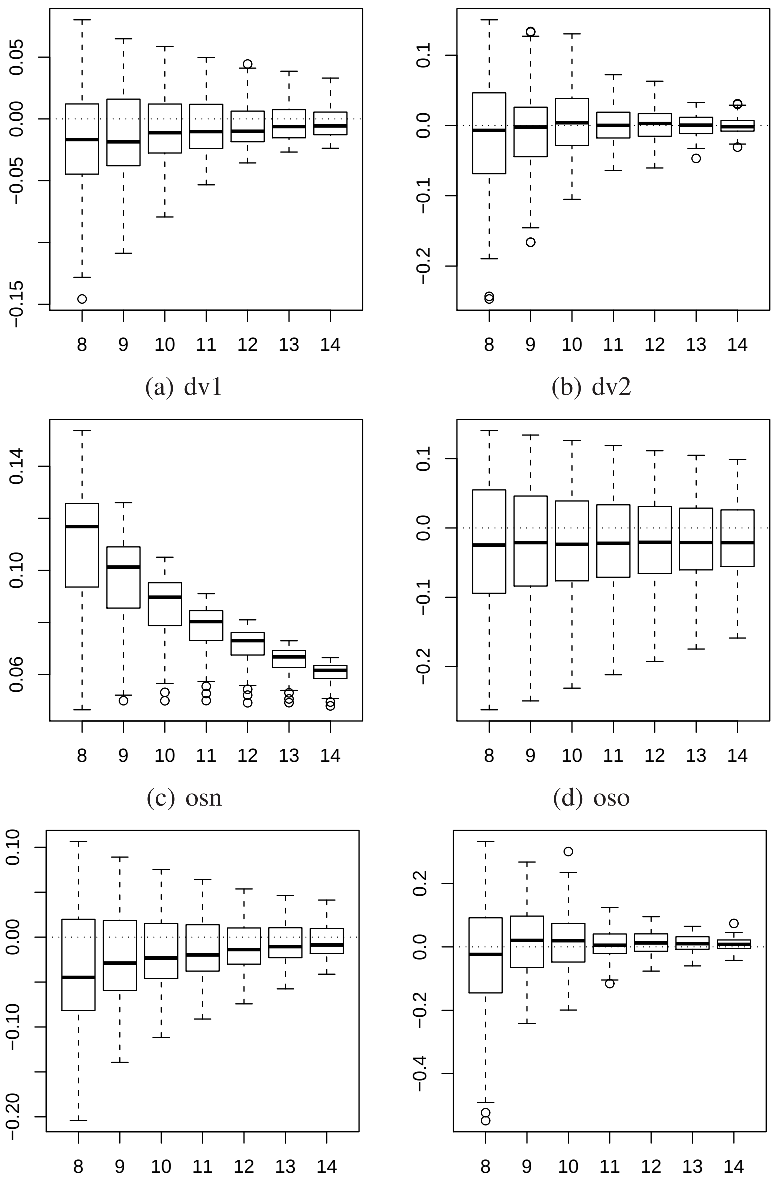
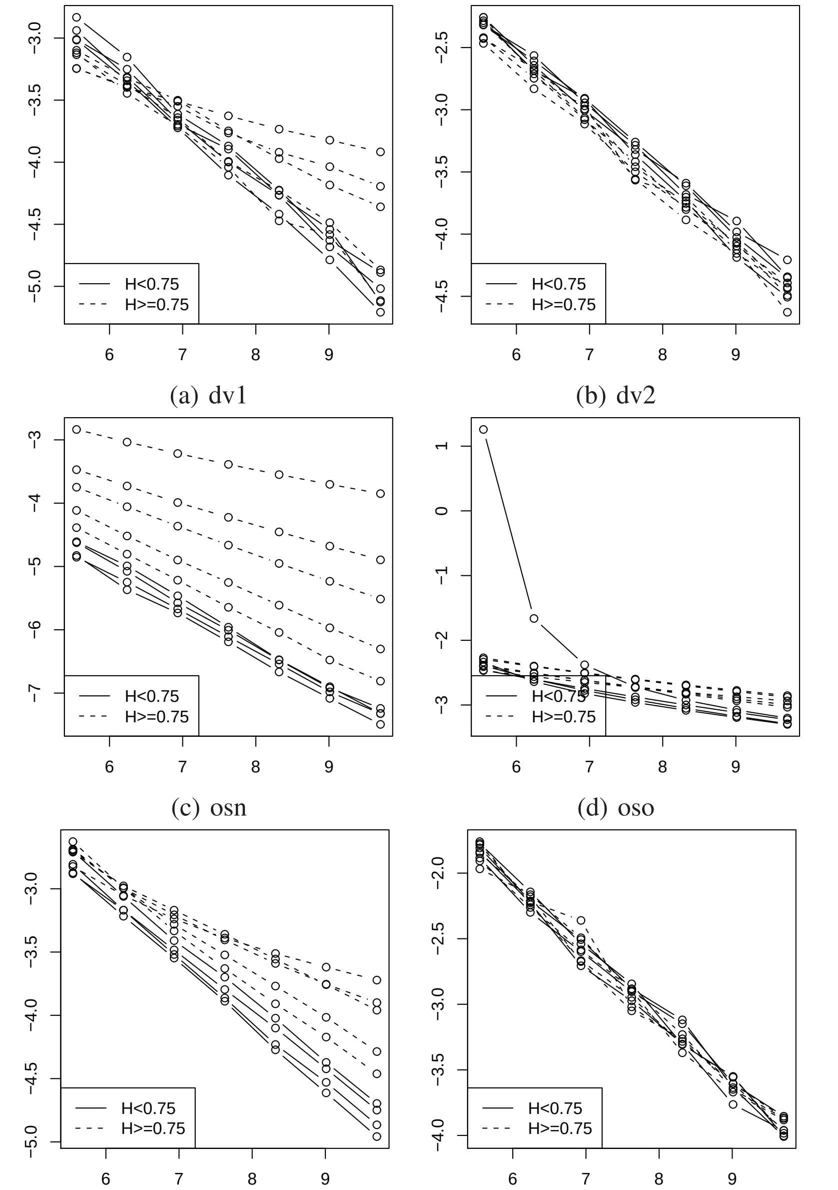


Fig. B.2. Boxplots for the O-U process, $H = 0.85$, $n = 2^k$, $k = 8, \ldots, 14$.



(a) dv1 (b) dv2

(c) osn (d) oso

(e) var (f) ir

Fig. B.3. Dependance of $log(SD)$ against $log(n)$ for the O-U process, $H \in \{0.55, 0.6, \ldots, 0.95\}$.

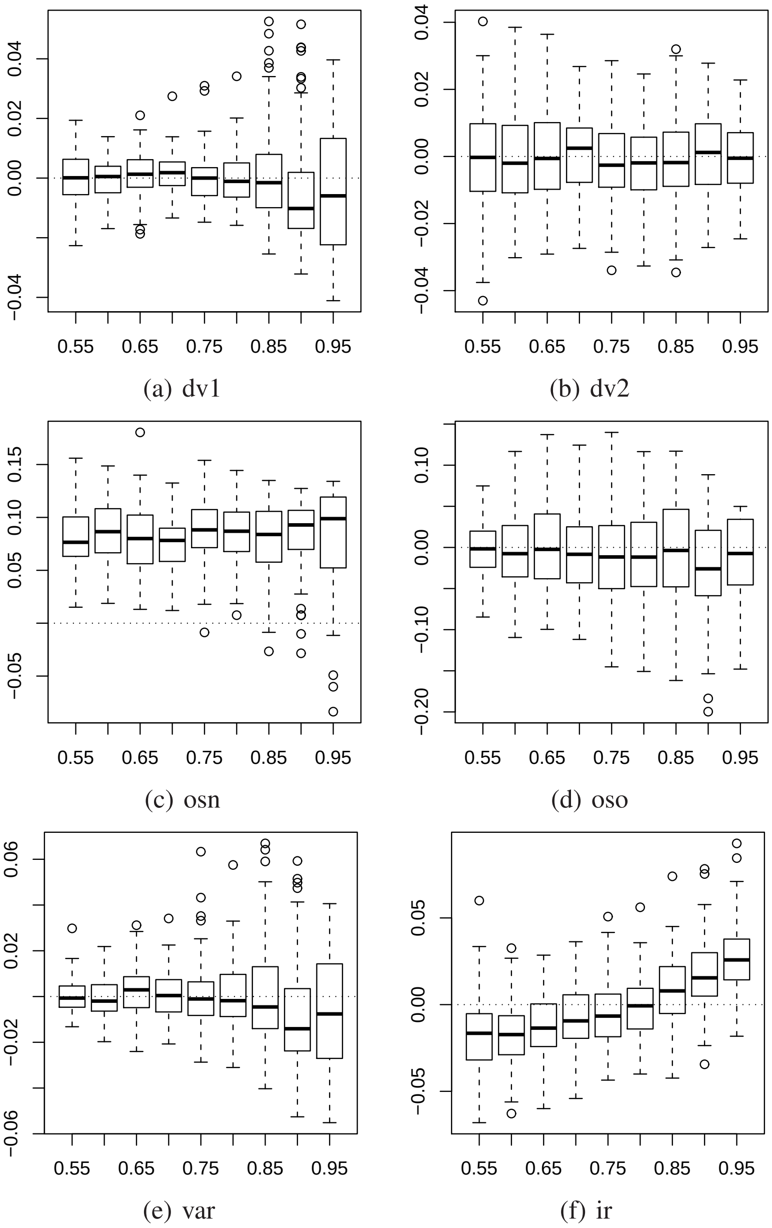


Fig. B.4. Boxplots for the B-S process, sample path length $n = 2^{14} + 1$.

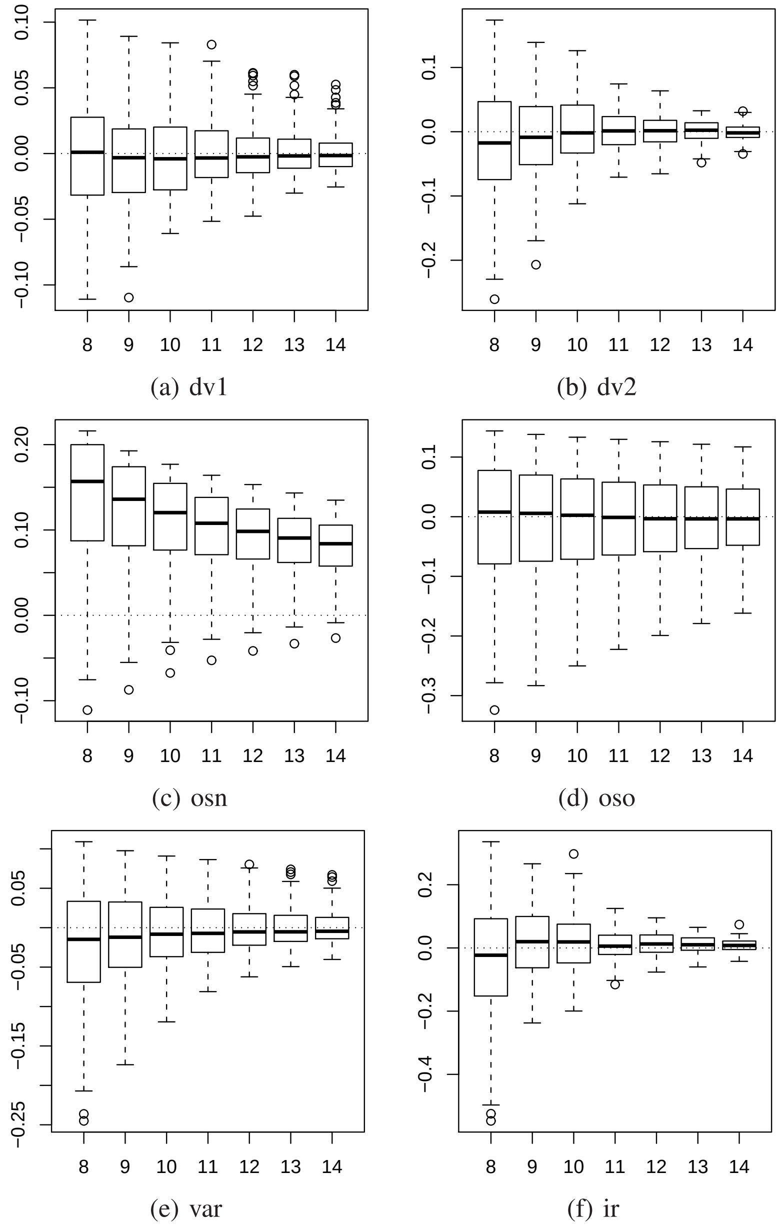
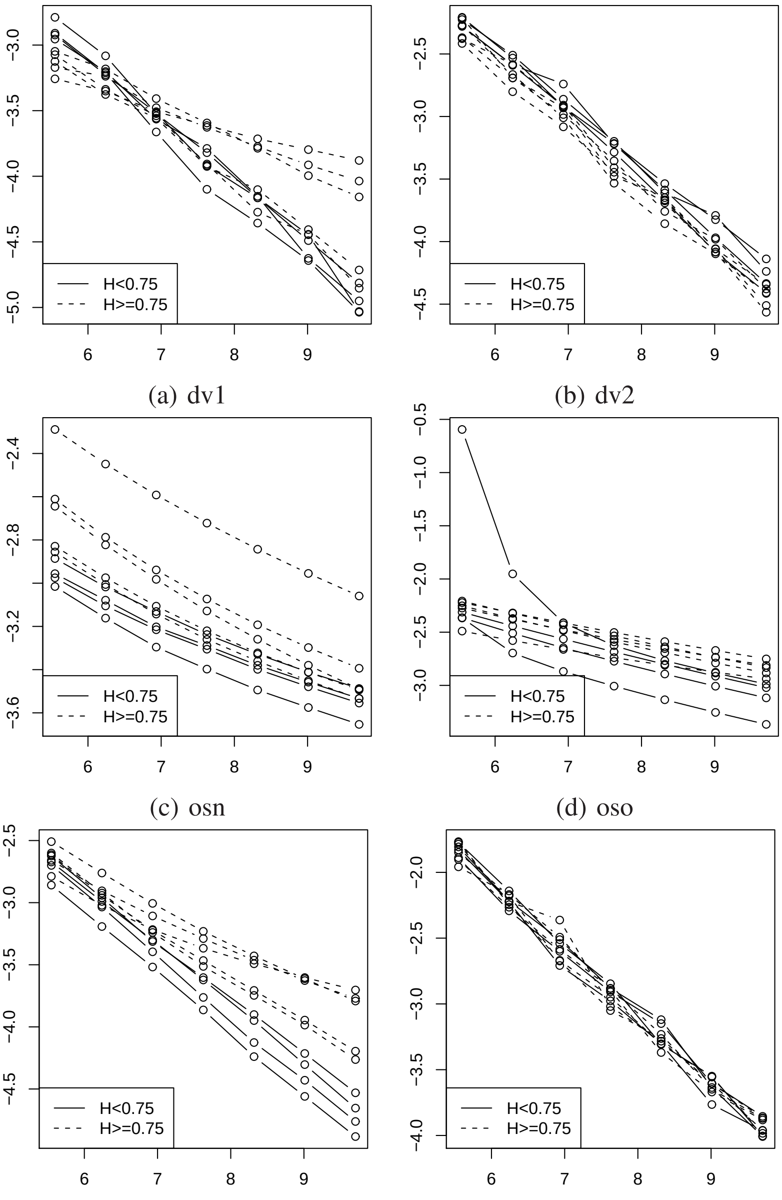


Fig. B.5. Boxplots for the B-S process, $H = 0.85$, $n = 2^k$, $k = 8, \ldots, 14$.



(a) dv1 (b) dv2

(c) osn (d) oso

(e) var (f) ir

Fig. B.6. Dependance of $log(SD)$ against $log(n)$ for the B-S process, $H \in \{0.55, 0.6, \ldots, 0.95\}$.

Dmitrij MELICHOV

ON ESTIMATION OF THE HURST INDEX
OF SOLUTIONS OF STOCHASTIC
DIFFERENTIAL EQUATIONS

Doctoral Dissertation
Physical Sciences, Mathematics (01P)

APIE STOCHASTINIŲ DIFERENCIALINIŲ
LYGČIŲ SPRENDINIŲ
HURSTO INDEKSO VERTINIMĄ

Daktaro disertacija
Fiziniai mokslai, Matematika (01P)

2011 11 03. 7,5 sp. l. Tiražas 20 egz.
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