

Nonparametric test for spatial geometric anisotropy

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Abstract. Paper deals with a problem of testing isotropy against geometric anisotropy for Gaussian spatial data. The original simple test statistic based on directional empirical semivariograms is proposed. Under the assumption of independence of the classical semivariogram estimators and for increasing domain asymptotics, the distribution of test statistics is approximated by chi-squared distribution. The simulation experiments demonstrate the efficacy of the proposed test.

Keywords: empirical semivariogram, anisotropy ratio, Gaussian random field.

Introduction

The assumption of spatial isotropy is often made in practice due to ease of computation and simpler interpretation. But in many applications spatial isotropy is not a reasonable assumption. A conventional practice when checking for isotropy is to assess plots of empirical semivariograms. However these graphical techniques are open to interpretation. Guan et al. [5] have proposed formal approach to test isotropy which is based on the asymptotic joint normality of empirical semivariograms for multiple directions. An L_2 -consistent subsampling estimator for asymptotic covariance matrix of the empirical semivariogram is used to construct a test statistic. But the subsampling procedure takes a large amount of computing time.

In the present paper we propose the simpler test statistic in Gaussian case under the assumption of independence of the classical semivariogram estimators.

1 Statistical models for spatial population

Suppose that spatial data are observations of a Gaussian random field (GRF) $\{Z(s): s \in D \subset R^m\}$ modeled by the equation

$$Z(s) = \mu + \varepsilon(s),$$

where μ constant mean and error term is zero-mean stationary GRF $\{\varepsilon(s): s \in D\}$.

The semivariogram of the GRF is defined by

$$\gamma(s - u) = \text{var} \{Z(s) - Z(u)\} / 2, \quad s, u \in D.$$

In many applications empirical evidence of directional effects are found in the semivariogram structure. The simplest way of dealing with anisotropic random fields

is to assume geometrical anisotropy. The geometric anisotropy means that models of the semivariogram have the same nugget, the same sill but different ranges in to perpendicular directions (see [5]). If one plots the directional ranges in 2D case they would fall on the edge of an ellipse, where major and minor axes of ellipse correspond to the largest and shortest ranges ($a_{\max}$ and $a_{\min}$) of directional semivariograms.

Algebraically, it adds to the isotropic model two more parameters: the anisotropy angle φ (angle made by major axis of ellipse and coordinate axis OY) and the anisotropy ratio

$$\lambda = \frac{a_{\max}}{a_{\min}} > 1. \quad (1)$$

Procedures of fitting of the geometric anisotropic semivariogram models to the environmental data can be easily realized by software system *R* package *Gstat* (see [1]). The geometric anisotropy refers to semivariogram of the form

$$\gamma(h) = \gamma_0([Ah]),$$

where γ_0 is an isotropic semivariogram with the range $a_{\max}$ and

$$A = \begin{pmatrix} \sin(\varphi) & \cos(\varphi) \\ -\lambda \cos(\varphi) & \lambda \sin(\varphi) \end{pmatrix}.$$

2 Empirical semivariogram and test for isotropy

In this paper we restrict our attention to the nuggetless model of covariance i.e.,

$$C(h) = \sigma^2 r(h),$$

where σ^2 is the variance (sill) and $r(h)$ is the spatial correlation function.

Denote by $S_n = \{s_i \in D; i = 1, \dots, n\}$ the set of locations where GRF $\{Z(s); s \in D\}$ is observed. The classical estimator of semivariogram is the method of moments estimator

$$\hat{\gamma}(h) = \frac{1}{2|N(h)|} \sum_{(s_i, s_j) \in N(h)} \{Z(s_i) - Z(s_j)\}^2. \quad (2)$$

Here $N(h)$ denotes all pairs (s_i, s_j) for which $s_i, s_j \in S_n$, $s_i - s_j = h$ and $|N(h)|$ denotes the cardinality of the set $N(h)$.

To test the hypothesis of isotropy, we choose the lag set A including spatial lags $h_1, h_2, \dots, h_K$ in the direction of major axis of ellipse and spatial lags $h_{K+1}, h_{K+2}, \dots, h_{2K}$ perpendicular to that direction. Assume that $|h_i| = |h_{i+K}|$, $i = 1, \dots, K$.

The hypothesis of isotropy is expressed as

$$H_0: \gamma(h_i) = \gamma(h_{i+K}), \quad i = 1, \dots, K.$$

Rejecting this hypothesis means accepting geometric anisotropy (hypothesis H_1).

Set $\Gamma' = (\gamma(h_1), \dots, \gamma(h_{2K}))$. Let $\hat{\Gamma} = (\hat{\gamma}(h_1), \dots, \hat{\gamma}(h_{2K}))$ be the vector of semivariogram estimators (2) obtained over S_n .

In what follows we establish the asymptotic properties of $\hat{\Gamma}$ under an increasing domain asymptotics, in which minimum distance between sampling points is bounded

away from zero and thus spatial domain of observation is unbounded. Under some regularity conditions, Guan et al. [3] proved that

$$\sqrt{n}(\hat{\Gamma} - \Gamma) \xrightarrow{D} N_{2K}(0, \Sigma_{\Gamma}) \quad \text{as } n \rightarrow \infty,$$

where Σ_{Γ} is the asymptotic covariance matrix with elements of complex structure.

Under the hypothesis of isotropy, there exists a full row rank matrix R such that $R\Gamma = 0$ [4]. Then under the hypothesis of isotropy it follows from continuous mapping theorem that

$$n(R\hat{\Gamma})'(R\Sigma_{\Gamma}R')^{-1}(R\hat{\Gamma}) \xrightarrow{D} \chi_r^2 \quad \text{as } n \rightarrow \infty, \quad (3)$$

where r denotes the row rank of R .

Following Cressie [2] we have

$$\text{var}(\hat{\Gamma}) \cong \text{diag}(2\gamma^2(h_1)/|N(h_1)|, \dots, 2\gamma^2(h_{2K})/|N(h_{2K})|),$$

where the approximation yields only little loss in estimation efficiency especially in the case of independence of the classical semivariogram estimators for different spatial lags.

We propose the following estimator of Σ_{Γ}

$$\hat{\Sigma}_{\Gamma} = n \text{diag}(2\hat{\gamma}^2(h_1)/|N(h_1)|, \dots, 2\hat{\gamma}^2(h_{2K})/|N(h_{2K})|),$$

and replace it in the statistic specified in (3) and form the test statistic

$$\hat{T} = n(R\hat{\Gamma})'(R\hat{\Sigma}_{\Gamma}R')^{-1}(R\hat{\Gamma}). \quad (4)$$

If H_0 is true, then chi-squared approximation for the distribution of the test statistic $\hat{T}$ is proposed.

So we suggest that an approximate size- p test for isotropy is to reject H_0 if $\hat{T} > \chi_{r,p}^2$, where $\chi_{r,p}^2$ is p -critical value of a chi-squared distribution with r degrees of freedom. Note that if H_1 is true, then the test statistic $\hat{T}$ can be approximated by linear combination of noncentral chi-squared random variables.

3 Simulation results

As an example we consider the case with D being integer regular 2-dimensional lattice. Set $h' = (h_x, h_y)$ for each $h \in D$. Simulations are done on 10×10 square grid. So the sample size is $n = 121$. We generated realizations from zero-mean, second-order stationary Gaussian random field. The case of geometric anisotropic spatial Gaussian correlation function $r(h) = \exp\{-(h_x^2 + \lambda^2 h_y^2)/\alpha^2\}$ is considered. Here α denotes the range parameter. For greater interpretability we consider the situation when the anisotropy angle φ is equal $\frac{\pi}{2}$ and $K = 2$, with $|h_1| = |h_3| = 1$, $|h_2| = |h_4| = 2$.

Set

$$R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Table 1. Empirical powers of test for simulated data.

λ	α					
	1	4	7	10	13	16
$M = 150$						
2	0.30	0.57	0.69	0.80	0.81	0.83
4	0.40	0.53	0.69	0.83	0.81	0.85
6	0.50	0.53	0.75	0.79	0.79	0.78
8	0.20	0.59	0.76	0.70	0.75	0.87
10	0.70	0.53	0.69	0.73	0.80	0.84
$M = 300$						
2	0.60	0.56	0.68	0.76	0.81	0.82
4	0.40	0.48	0.72	0.77	0.77	0.84
6	0.60	0.58	0.71	0.76	0.77	0.84
8	0.50	0.53	0.67	0.76	0.78	0.82
10	0.50	0.51	0.69	0.78	0.80	0.78
$M = 600$						
2	0.50	0.57	0.71	0.76	0.80	0.83
4	0.30	0.54	0.71	0.75	0.78	0.81
6	0.40	0.53	0.73	0.76	0.82	0.83
8	0.50	0.54	0.72	0.76	0.80	0.81
10	0.50	0.52	0.67	0.79	0.80	0.84

Then the test statistic specified in (4) is

$$\hat{T} = \frac{1}{2} \sum_{i=1}^2 (\hat{\gamma}(h_i) - \hat{\gamma}(h_{i+2}))^2 / (\hat{\gamma}^2(h_i) / |N(h_i)| + \hat{\gamma}^2(h_{i+2}) / |N(h_{i+2})|).$$

Its approximate distribution is the χ_r^2 distribution. As a performance measure of the proposed test statistic we considered the empirical power of test (frequency of rejecting H_0 for simulated geometrically anisotropic Gaussian data) with significance level $p = 0.05$. For various values of the anisotropy ratio λ specified in (1) and the range parameters α , simulations with three different numbers of replications M are performed. Empirical powers of test for three simulated data sets with $M = 150, 300, 600$ are presented in Table 1.

Table 1 shows that empirical power of test increases with increasing of range parameter, but empirical power is not influenced by the anisotropy ratio. So we propose to use our test statistic for the particular cases of geometrically anisotropic spatial Gaussian data.

References

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REZIUUMÉ

Neparametrinis testas erdvinių duomenų geometrinei anizotropijai nustatyti.*K. Dučinskas, L. Drežienė*

Straipsnyje pasiūlytas paprastas testas erdvinių Gauso duomenų geometrinei anizotropijai nustatyti. Testo statistikos skirstinys aproksimuojamas χ_r^2 skirstiniu. Pavyzdyje parodytas didelis šio testo galingumas stipriai koreliuotiems Gauso duomenims.

Raktiniai žodžiai: empirinė semivariograma, anizotropijos santykis, atsitiktinis Gauso laukas.