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A Beveridge–Nelson filters for the self normalization *

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Abstract. Let $X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$ be a linear process, where ε_t, $t \in Z$, are i.i.d. r.v.'s in the domain of attraction of a normal law with zero mean and possibly infinite variance. Generalizing the class of Beveridge–Nelson filters this article proves a central limit theorem for the self-normalized sums $U_n^{-1} \sum_{t=1}^{n} X_t$, where U_n^2 is a sum of squares of block-sums of size m, as m and the number of blocks $N = n/m$ tend to infinity.

Keywords: linear process, normal law.

1. Introduction

Since the work of Peter C.B. Phillips and V. Solo [3] a method of deriving asymptotics for weakly dependent linear processes has been used with an explicit algebraic decomposition of the linear filter. The method offers a simple unified approach to strong laws, central limit theorem and invariance principles for w. d. linear processes. In the article [2] Juodis and Račkauskas proves self-normalized central limit theorem for the Beveridge–Nelson [1] linear processes. In this paper we generalize this theorem allowing to widen the class of filters. We deal with the linear process of the following form

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \psi_j^* = \sum_{i=j}^{\infty} \psi_i. \tag{1}$$

In this section we consider filters satisfying the main summation condition

$$\sum_{j=0}^{\infty} (\psi_j^*)^2 < \infty. \tag{2}$$

This condition is important in the sense that the Beveridge–Nelson remains must be stationary. Now we are ready to define a new class of linear filters

$$\Gamma_\psi := \bigcup_{p<2} \Big\{ (\psi_i)_{i \geqslant 0} : \sum_{k=0}^{\infty} k^p |\psi_k|^p < \infty \Big\}. \tag{3}$$

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Observe that this class is smaller than

$$\sum_{k=0}^{\infty} k^2 |\psi_k|^2 < \infty. \tag{4}$$

But the difference between them is measured only by the slowly varying factor.

An important class of linear processes is the so called Hall-Heyde condition

$$\sum_{k=0}^{\infty} k^{1/2} |\psi_k| < \infty. \tag{5}$$

Observe that both of them imply (2) but they differs from each other. Indeed filters $\psi_j = 1/j^{3/2} \ln(j+1)$ satisfies (4), but fails on (5). The essence that all classes mentioned above are of order $3/2$, and they only can differ by slowly varying factor. And even the class (2) can't afford the filters $1/j^{3/2}$.

The classical B-N case considers the filters

$$\sum_{k=0}^{\infty} k |\psi_k| < \infty. \tag{6}$$

This condition allows classical Beveridge–Nelson decomposition, but it is very strong(i.e., filters $\psi_j = 1/j^2$ fails to satisfy it).

2. Results

Let X_t be a linear process of the form (1), where ε_t, $t \in Z$, are independent identically distributed random variables in the domain of attraction of a normal law with zero mean and possibly infinite variance (i.e., there exists constants b_n such that $b_n^{-1}(X_1 + \ldots + X_n) \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1)$ denoted $X_i \in DAN$, here and throughout $\xrightarrow[n\to\infty]{\mathcal{D}}$ means weak convergence, and $N(0,1)$ standard normal law). Denote $S_n = X_1 + \ldots + X_n$. Set $U_0 = 0$ and define

$$U_n := U_{m,k}^2 = \sum_{j=1}^{k} \left(S_{jm} - S_{(j-1)m}\right)^2, \quad k = 1, \ldots, N, \ \ 1 \leqslant m < n, \tag{7}$$

where $N = [n/m]$ and $[a]$ denotes the integer part of a. In the paper [2] the main result is Theorem 2 which states

$$U_n^{-1} S_n \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1), \tag{8}$$

for the class (6). The following theorem is the main result of this paper, and actually it generalizes Juodis Račkauskas Theorem 2 for the bigger class Γ_ψ.

THEOREM 1. *If $(\psi_n) \in \Gamma_\psi$, and $\varepsilon_i \in DAN$, $E\varepsilon_i = 0$, then*

$$U_n^{-1} S_n \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1).$$

Proof. If we check the proof of the Theorem 2 [2] we see that we only need to re-estate the following statements

$$(V_n^{\varepsilon})^{-2}\sum_{j=1}^{N}\Big(\sum_{k=0}^{jm-1}\psi_k^*\varepsilon_{jm-k}\Big)^2 \to^P 0 \tag{9}$$

and

$$(V_n^{\varepsilon})^{-2}\sum_{j=1}^{N}\Big(\sum_{k=0}^{\infty}\big(\psi_{jm+k}^*-\psi_{(j-1)m+k}^*\big)\varepsilon_{-k}\Big)^2 \to^P 0. \tag{10}$$

Observe that by stationarity of the process $(\varepsilon_k/V_n^{\varepsilon}, k=1,\dots,n)$ we have

$$E(V_n^{\varepsilon})^{-2}\sum_{j=1}^{N}\Big(\sum_{k=0}^{jm-1}\psi_k^*\varepsilon_{jm-k}\Big)^2 \leqslant \frac{N}{n}\sum_{k=0}^{\infty}(\psi_k^*)^2+\frac{CN}{n^2}\Big(\sum_{k=0}^{N}|\psi_k^*|\Big)^2$$

and by Jensen's inequality one has

$$\leqslant \frac{N}{n}\sum_{k=0}^{\infty}(\psi_k^*)^2+\frac{CN^2}{n^2}\sum_{k=0}^{\infty}(\psi_k^*)^2.$$

Thus condition (2) is sufficient for the (9) convergence. Recall that

$$\Gamma_N^2=\sum_{j=1}^{N}\Big(\sum_{k=0}^{\infty}\big(\psi_{jm+k}^*-\psi_{(j-1)m+k}^*\big)\varepsilon_{-k}\Big)^2.$$

Now convergence (10) reduces in showing that Γ_N is stochastically bounded. To this aim we need seven steps. First observe that

$$|\Gamma_N|^p \leqslant \sum_{j=1}^{N}\Big|\sum_{k=0}^{\infty}\big(\psi_{jm+k}^*-\psi_{(j-1)m+k}^*\big)\varepsilon_{-k}\Big|^p.$$

Next we use the moment inequality

$$E\Big|\sum\xi_i\Big|^p \leqslant 2\sum E|\xi_i|^p, \tag{11}$$

which is true for any r.v.'s ξ_i if $0<p\leqslant 1$ and for any martingale differences ξ_i's if $1<p\leqslant 2$.

Hence

$$E|\Gamma_n|^p \leqslant 2E|\varepsilon_1|^p\sum_{j=1}^{N}\sum_{k=0}^{\infty}\Big|\sum_{i=k+(j-1)m+1}^{k+jm}\psi_i\Big|^p.$$

Third we interchange the summation order

$$= 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \sum_{j=1}^{N} \Big| \sum_{i=k+(j-1)m+1}^{k+jm} \psi_i \Big|^p.$$

Fourth step

$$\leqslant 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \sum_{j=1}^{N} \Big(\sum_{i=k+(j-1)m+1}^{k+jm} |\psi_i| \Big)^p.$$

Since $(a_1^p + \cdots + a_N^p)^{1/p}$ is monotonically decreasing, thus

$$\leqslant 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \Big(\sum_{i=k+1}^{k+n} |\psi_i| \Big)^p.$$

Next step

$$\leqslant 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \Big(\sum_{i=k+1}^{\infty} |\psi_i| \Big)^p.$$

And finally the seventh step we use [3] page 987 top inequality

$$\leqslant 2E|\varepsilon_1|^p \Big(const \cdot \sum_{k=0}^{\infty} k^p |\psi_k|^p \Big).$$

Hence

$$E|\Gamma_n|^p \leqslant E|\varepsilon_1|^p \sum_{j=1}^{N} \Big| \sum_{k=0}^{\infty} \sum_{i=k+(j-1)m+1}^{k+jm} |\psi_i| \Big|^p \leqslant cE|\varepsilon_1|^p \sum_{k=0}^{\infty} k^p |\psi_k|^p.$$

Thus the proof for generalized clt for Γ_ψ class is completed.

References

1. S. Beveridge, C.R. Nelson, A new approach to decomposition of economic time series into permanent and transitory components with particular attention to measurement of the 'buiseness cycle', *Journal of Monetary Economics*, **7**, 151–174 (1981).
2. M. Juodis, A. Račkauskas, A central limit theorem for self-normalized sums of a linear process, *Statistics Probability Letters*, **77**, 1535–1541 (2007).
3. P.C.B. Phillips, V. Solo, Asymptotics for linear processes, *Ann. Stat.*, **20**, 971–1001 (1992).

REZIUMĖ

M. Juodis. Beveridge–Nelson filtrų apibendrinimai autonormuotoms sumoms

Darbe įrodoma blokais autonormuota centrinė ribinė teorema. Nagrinėjami tiesiniai procesai su apibendrintu Beveridge–Nelson filtru.