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Investigation of Negative Critical Points of the Characteristic Function for Problems with Nonlocal Boundary Conditions

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Abstract. In this paper the Sturm-Liouville problem with one classical and the other nonlocal two-point or integral boundary condition is investigated. Critical points of the characteristic function are analyzed. We investigate how distribution of the critical points depends on nonlocal boundary condition parameters. In the first part of this paper we investigate the case of negative critical points.

Keywords: Sturm-Liouville problem, nonlocal boundary condition, critical points.

1 Introduction

Differential problems with nonlocal boundary conditions arise in various fields of biology, biotechnology, physics, etc. Theoretical investigation of problems with different types of nonlocal boundary conditions is a topical problem and recently much attention has been paid to them in the scientific literature. The analysis of eigenvalues of the difference operator with a nonlocal condition permits us to investigate the stability of difference schemes and corroborate the convergence of iterative methods [1–5] and it is also of interest in itself. Eigenvalues and eigenfunctions of differential problems with nonlocal two-point boundary conditions are investigated by A.V. Gulin and V. A. Morozova [6], N. I. Ionkin and E. A. Valikova [7], M. Sapagovas and A. Štikonas [8], Štikonas [9], S. Pečiulytė [10–13]. Such problems with nonlocal integral boundary conditions are analyzed B. Bandyrskii, I. Lazurchak, V. Makarov and M. Sapagovas [14], R. Čiupaila, Ž. Jesevičiūtė and M. Sapagovas [15], G. Infante [16], A. Štikonas and S. Pečiulytė [10, 17], etc. In recent decades the number of differential problems with nonlocal boundary conditions and numerical methods for such problems have increased significantly.

Investigation of the spectra of differential equations with nonlocal conditions is quite a new area related to the problems of this type. In this paper, we investigate the Sturm-Liouville problem with a classical the first type boundary condition on the left side of the interval (also the second type boundary condition) and with four cases of nonlocal two-point and two cases of nonlocal integral boundary conditions on the right boundary. We analyze critical and other points of a real characteristic function.

For the fixed parameter ξ, dependence of spectra of these problems on the parameter γ in nonlocal boundary conditions has been investigated in the previous research (see, [8, 11, 13, 15, 17]) and S. Pečiulytė Doctoral Thesis [10]. Furthermore, conditions, where there exist constant, negative and only real eigenvalues have been obtained in these articles. The first results on the dependence of distribution constant and critical points on the parameter $\xi \in (0, 1)$ were presented in [12]. We extend here our investigation and the new results on the critical points of the characteristic functions are presented.

In the first part of this paper we formulate a few problems with nonlocal boundary conditions in Section 2. Then we give a definition of a real characteristic function in Section 3, we find zeroes, poles and constant eigenvalue points for this function and investigate critical points in Section 4. The distribution of negative critical points are presented in Section 5.

2 Some problems with nonlocal boundary conditions

Let us analyze the Sturm-Liouville problem with one classical boundary condition

$$-u'' = \lambda u, \quad t \in (0, 1), \tag{1}$$

$$u(0) = 0, \tag{2}$$

and the other nonlocal two-point boundary condition of the Samarskii-Bitsadze type or integral type:

$$u'(1) = \gamma u(\xi), \qquad \text{(Case 1)} \qquad (3_1)$$

$$u'(1) = \gamma u'(\xi), \qquad \text{(Case 2)} \qquad (3_2)$$

$$u(1) = \gamma u'(\xi), \qquad \text{(Case 3)} \qquad (3_3)$$

$$u(1) = \gamma u(\xi), \qquad \text{(Case 4)} \qquad (3_4)$$

$$u(1) = \gamma \int_0^\xi u(t)\mathrm{d}t, \qquad \text{(Case 5)} \qquad (3_5)$$

$$u(1) = \gamma \int_\xi^1 u(t)\mathrm{d}t, \qquad \text{(Case 6)} \qquad (3_6)$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0, 1]$. Also, we analyze the Sturm-Liouville problem (1) with the boundary condition

$$u'(0) = 0 \tag{4}$$

on the left side and with nonlocal boundary conditions (3) on the right side of the interval. We enumerate these cases from Case 1′ to Case 6′, respectively. We denote problems (1), (2) in the case of nonlocal boundary conditions (3_1)–(3_6) as P1, P2, P3, P4, P5, P6 and problems (1), (4) in the case of nonlocal boundary conditions (3_1)–(3_6) as P1′, P2′, P3′, P4′, P5′, P6′, respectively. Note that the index in the references denotes the case. If there is no index, then the rule (or results) holds on in all the cases of nonlocal boundary conditions.

Remark 1 (Classical case)**.** *We have the classical case for* $\gamma = 0$*. Eigenvalues in this case are well known:*

$$\lambda_k^{cl} = k^2\pi^2, \qquad u_k(t) = \sin(k\pi t), \qquad k \in \mathbb{N}; \tag{$5_{3,4,5,6}$}$$

$$\lambda_k^{cl} = (k-1/2)^2\pi^2, \quad u_k(t) = \sin\big((k-1/2)\pi t\big), \quad k \in \mathbb{N}; \tag{$5_{1,2}$}$$

$$\lambda_k^{cl} = (k-1/2)^2\pi^2, \quad u_k(t) = \cos\big((k-1/2)\pi t\big), \quad k \in \mathbb{N}; \tag{$5_{3',4',5',6'}$}$$

$$\lambda_k^{cl} = (k-1)^2\pi^2, \qquad u_k(t) = \cos\big((k-1)\pi t\big), \qquad k \in \mathbb{N}. \tag{$5_{1',2'}$}$$

We get the same case for $\xi = 0$ *(Problems* P1, P4, P5, P2′, P3′, P5′*),* $\xi = 1$ *(Problems* P6, P6′*),* $\xi = 1$ *and* $\gamma \neq 1$ *(Problems* P2, P4, P2′, P4′*). In the case* $\xi = 1$ *and* $\gamma = 1$ *(Problems* P2, P4, P2′, P4′*) we have degenerate case with one left boundary condition. So, we omit these cases and define* $D_\xi := [0,1]$ *(Problems* P3, P1′*),* $D_\xi := (0,1]$ *(Problems* P1, P5, P3′, P5′*),* $D_\xi := [0,1)$ *(Problems* P2, P6, P4′, P6′*),* $D_\xi := (0,1)$ *(Problems* P4, P2′*).*

Remark 2 (Case $\gamma = \infty$)**.** *In this case, we define boundary conditions:* $u(\xi) = 0$*,* $u'(\xi) = 0$*,* $u'(\xi) = 0$*,* $u(\xi) = 0$*,* $\int_0^\xi u(t)dt = 0$*,* $\int_\xi^1 u(t)dt = 0$*, accordingly.*

Firstly, let us consider the case where ξ is fixed. We define a *constant eigenvalue* as the eigenvalue $\lambda = q^2$ that does not depend on the parameter $\gamma \in \overline{\mathbb{C}}$ [11, 17]. For any constant eigenvalue we define the *constant eigenvalue point* $q \in \mathbb{C}_q := \{z \in \mathbb{C}\colon -\pi/2 < \arg z \leqslant \pi/2 \text{ or } z = 0\}$ and the *constant eigenvalue* γ*-value point* $(q,\gamma) \in \mathbb{C}_q \times \overline{\mathbb{C}}$, respectively. For a constant eigenvalue, the set of γ-value points in $\mathbb{C}_q \times \overline{\mathbb{C}}$ is a vertical line. Other eigenvalues will be named as *nonconstant*. For such eigenvalues, we define a nonconstant eigenvalue point $q(\gamma) \in \mathbb{C}_q$ and a nonconstant eigenvalue γ-point $(q, \gamma(q)) \in \mathbb{C}_q \times \overline{\mathbb{C}}$. In the nonconstant eigenvalue case, we get eigenvalue points as roots of the equation $f_1(q) - \gamma f_2(q) = 0$, where $f_1(q) := \sin q/q$ for Problems P1–P6, and $f_1(q) := \cos q$ for Problems P1′–P6′. The function $f_2(q)$ depends on the case of the second boundary condition. We get all the constant eigenvalue points by solving the system

$$\begin{cases} f_1(q) = 0, \\ f_2(q) = 0. \end{cases}$$

Corollary 1. *The point* $q = 0$ *cannot be a constant eigenvalue point for problems* (1) *with boundary conditions* (2) *or* (4)*.*

All nonconstant eigenvalues (which depend on the parameter γ) are γ-points of the meromorphic functions $\gamma_c = \gamma_c(q) = f_1(q)/f_2(q)\colon \mathbb{C}_q \to \overline{\mathbb{C}}$. We call this function γ_c as a *complex characteristic function*.

We enumerate the eigenvalues $\lambda_k = \lambda_k(\gamma, \xi)$ using the classical case $\gamma = 0$, i.e., $\lambda_k(0, \xi) = \lambda_k^{\mathrm{cl}}$. The eigenvalues λ_k (and eigenvalue points $q_k := \sqrt{\lambda_k}$) depend on the parameter γ continuously. All zeroes (zeroes of the function $f_1(q)$) and poles (zeroes of the function $f_2(q)$) of the complex characteristic function for investigated problems (see, [10–13]) are nonnegative real numbers. If a zero of the function $f_1(q)$ is coincident with a pole, i.e., a zero of the function $f_2(q)$, then this point is a constant eigenvalue point. In fact we must find the set $\mathcal{Z}$ of all zeroes of the real characteristic function γ. Then the set of the constant eigenvalue points $\mathcal{C} = \{k\pi, k \in \mathbb{N}\} \setminus \mathcal{Z}$.

We call the point $q_c \in \mathbb{C}_q$, $q_c \neq 0$ such that $\gamma_c'(q_c) = 0$, a *critical point* of the complex characteristic function, and we call an image of the critical point $\gamma_c(q_c)$ *a critical value* of the complex characteristic function [12].

3 Real characteristic function

If we take q only in the rays $q = x \geqslant 0,\ q = -\mathrm{i}x,\ x \leqslant 0$ instead of $q \in \mathbb{C}_q$, we investigate positive eigenvalues in case the ray $q = x > 0$, and we get negative eigenvalues in the ray $q = -x,\ x < 0$. The point $q = x = 0$ corresponds to $\lambda = 0$. We have two restrictions of the function $\gamma_c\colon \mathbb{C}_q \to \mathbb{R}$ on those rays: $\gamma_+(x) := \gamma_c(x + \mathrm{i}0)$ for $x \geqslant 0$ and $\gamma_-(x) := \gamma_c(0 - \mathrm{i}x)$ for $x \leqslant 0$. The function γ_+ corresponds to the case of positive eigenvalues, while the function γ_- to that of negative eigenvalues. All the real eigenvalues

$$\lambda_k = \begin{cases} x_k^2 & \text{for } x_k \geqslant 0, \\ -x_k^2 & \text{for } x_k \leqslant 0, \end{cases} \quad k \in \mathbb{N}, \tag{6}$$

can be investigated using a *real characteristic function* $\gamma\colon \mathbb{R} \to \mathbb{R}$ (see, [11, 17]):

$$\gamma(x) = \begin{cases} \gamma_+(x) = \gamma_c(x) & \text{for } x \geqslant 0, \\ \gamma_-(x) = \gamma_c(-\mathrm{i}x) & \text{for } x \leqslant 0. \end{cases}$$

Let us write an expression of the characteristic function in each case of the nonlocal boundary condition [10, 11, 17] for $\xi \in D_\xi$:

$$\gamma = \frac{1}{\xi} \cdot \frac{f(x)}{g(\xi x)}, \quad \begin{cases} f(x) := \cosh x, & g(x) := \frac{\sinh x}{x} \quad \text{for } x \leqslant 0, \\ f(x) := \cos x, & g(x) := \frac{\sinh x}{x} \quad \text{for } x \geqslant 0; \end{cases} \tag{$7_{1,5'}$}$$

$$\gamma = \frac{f(x)}{g(\xi x)}, \quad \begin{cases} f(x) := \cosh x, & g(x) := \cosh x \quad \text{for } x \leqslant 0, \\ f(x) := \cos x, & g(x) := \cos x \quad \text{for } x \geqslant 0; \end{cases} \tag{$7_{2,4'}$}$$

$$\gamma = \frac{f(x)}{g(\xi x)}, \quad \begin{cases} f(x) := \frac{\sinh x}{x}, & g(x) := \cosh x \quad \text{for } x \leqslant 0, \\ f(x) := \frac{\sin x}{x}, & g(x) := \cos x \quad \text{for } x \geqslant 0; \end{cases} \tag{7_3}$$

$$\gamma = \frac{1}{\xi}\cdot\frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \frac{\sinh x}{x}, & g(x) := \frac{\sinh x}{x} \quad \text{for } x \leqslant 0, \\ f(x) := \frac{\sin x}{x}, & g(x) := \frac{\sin x}{x} \quad \text{for } x \geqslant 0; \end{cases} \tag{$7_{4,2'}$}$$

$$\gamma = \frac{2}{\xi^2}\cdot\frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \frac{\sinh x}{x}, & g(x) := \frac{\cosh x - 1}{x^2/2} \quad \text{for } x \leqslant 0, \\ f(x) := \frac{\sin x}{x}, & g(x) := \frac{1-\cos x}{x^2/2} \quad \text{for } x \geqslant 0; \end{cases} \tag{7_5}$$

$$\gamma = \frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := x\sinh x, & g(x) := \cosh x \quad \text{for } x \leqslant 0, \\ f(x) := -x\sin x, & g(x) := \cos x \quad \text{for } x \geqslant 0; \end{cases} \tag{$7_{1'}$}$$

$$\gamma = \xi\cdot\frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \cosh x, & g(x) := x\sinh x \quad \text{for } x \leqslant 0, \\ f(x) := \cos x, & g(x) := -x\sin x \quad \text{for } x \geqslant 0; \end{cases} \tag{$7_{3'}$}$$

$$\gamma = \frac{2}{1-\xi^2}\cdot\frac{f(x)}{g(\frac{1+\xi}{2}x)g(\frac{1-\xi}{2}x)} = \begin{cases} \frac{x\sinh x}{\cosh(\xi x)-1} & \text{for } x \leqslant 0, \\ \frac{x\sin x}{1-\cos(\xi x)} & \text{for } x \geqslant 0, \end{cases}$$

$$\begin{cases} f(x) := \frac{\sinh x}{x}, & g(x) := \frac{\sinh x}{x} \quad \text{for } x \leqslant 0, \\ f(x) := \frac{\sin x}{x}, & g(x) := \frac{\sin x}{x} \quad \text{for } x \geqslant 0; \end{cases} \tag{7_6}$$

$$\gamma = \frac{1}{1-\xi}\cdot\frac{f(x)}{g_1(\frac{1+\xi}{2}x)g_2(\frac{1-\xi}{2}x)} = \begin{cases} \frac{x\cosh x}{\sinh x-\sinh(\xi x)} & \text{for } x \leqslant 0, \\ \frac{x\cos x}{\sin x-\sin(\xi x)} & \text{for } x \geqslant 0, \end{cases}$$

$$\begin{cases} f(x) := \cosh x, & g_1(x) := \frac{\sinh x}{x}, & g_2(x) := \cosh x \quad \text{for } x \leqslant 0, \\ f(x) := \cos x, & g_1(x) := \frac{\sin x}{x}, & g_2(x) := \cos x \quad \text{for } x \geqslant 0. \end{cases} \tag{$7_{6'}$}$$

Characteristic functions are the same for the problems P1 and P5′, P2 and P4′, P4 and P2′, accordingly. Thus, these problems have the same spectrum.

Now we formulate obvious properties of the functions f, g, g_1, g_2 as following proposition. Some of these properties were investigated in [10, 11, 17].

Proposition 1. *The point $z_0 = 0$ is zero of the second order for the function f in Case $1'$, and the points $p_k = 2\pi k$ are zeroes of the second order for the function g in Case 5 for $k \in \mathbb{N}$ and in Case $3'$ for $k = 0$:*

$$f(z_0) = f'(z_0) = 0, \;\; f''(z_0) \neq 0, \quad g(p_k) = g'(p_k) = 0, \;\; g''(p_k) \neq 0. \tag{8}$$

Other zero points z of the functions $f(x)$, $g(x)$, $g_1(x)$, $g_2(x)$ are of the first order

$$f(z) = 0, \;\; f'(z) \neq 0, \;\; g(z) = 0, \;\; g'(z) \neq 0, \;\; g_i(z) = 0, \;\; g_i'(z) \neq 0, \;\; i = 1,2. \tag{9}$$

These positive zeroes of the first order of the function f are equal to:

$$z_k := (k - 1/2)\pi, \quad k \in \mathbb{N}, \tag{$10_{1,2,3',4',5',6'}$}$$

$$z_k := k\pi, \quad k \in \mathbb{N}; \tag{$10_{3,4,5,1',2',6}$}$$

the positive zeroes of the first order of the function g are equal to:

$$\tilde{p}_k := (k - 1/2)\pi, \quad k \in \mathbb{N}, \tag{$11_{2,3,1',4'}$}$$

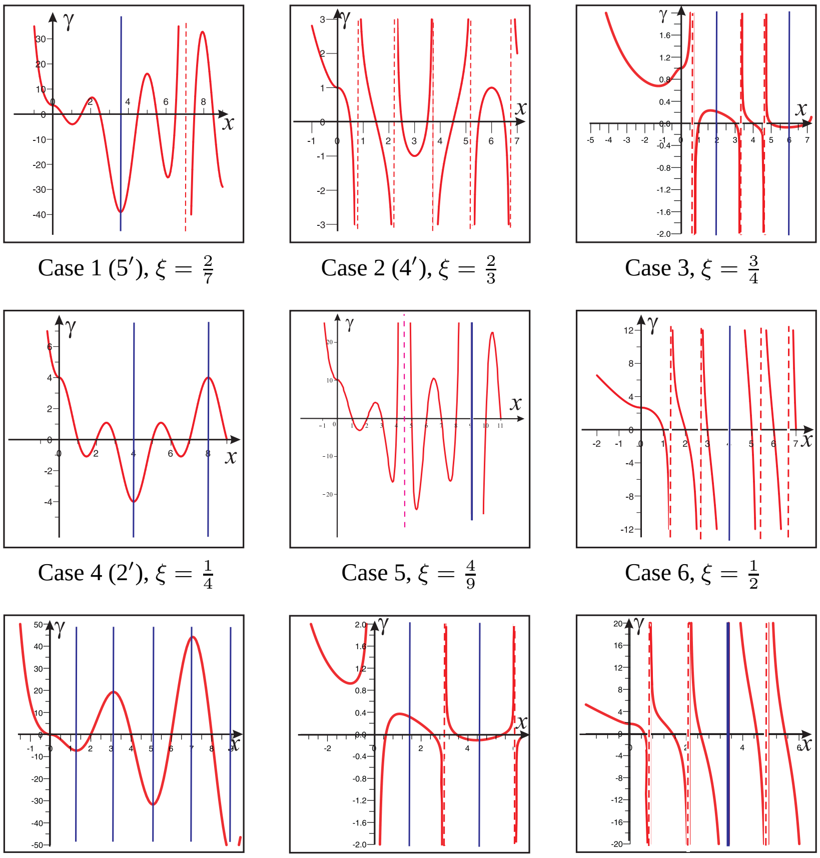
$$\tilde{p}_k := k\pi, \quad k \in \mathbb{N}; \tag{$11_{1,4,2',3',5',6}$}$$

the positive zeroes of the first order are equal to:

$$\tilde{p}_k := k\pi, \ k \in \mathbb{N} \ \textit{for} \ g_1, \qquad \tilde{p}_k := (k - 1/2)\pi, \ k \in \mathbb{N} \ \textit{for} \ g_2; \tag{$12_{6'}$}$$

and there are no zeroes of the first order in Case 5*.*

The graphs of characteristic functions for some ξ are presented in Fig. 1. The vertical solid lines correspond to constant eigenvalues, vertical dashed lines cross the x-axis at points of poles. For some cases the vertical line of the constant eigenvalue coincides with the vertical asymptotic line at the point of a pole.



Case 1 (5′), $\xi = \frac{2}{7}$ Case 2 (4′), $\xi = \frac{2}{3}$ Case 3, $\xi = \frac{3}{4}$

Case 4 (2′), $\xi = \frac{1}{4}$ Case 5, $\xi = \frac{4}{9}$ Case 6, $\xi = \frac{1}{2}$

Case 1′, $\xi = \frac{1}{2}$ Case 3′, $\xi = \frac{2}{3}$ Case 6′, $\xi = \frac{3}{7}$

Fig. 1. Real characteristic functions $\gamma_l(x\pi)$.

Remark 3. *Note, that the positive part of the x-axis is scaled π times and $x = 1$ is really $x = \pi$ in all figures.*

3.1 Zeroes, poles and constant eigenvalues points

The characteristic function has the zero point z, if $f(z) = 0$ and $g(\xi z) \neq 0$ (Problems P1–P5, P1′–P5′), $g(\frac{1+\xi}{2}z) \neq 0$ or $g(\frac{1-\xi}{2}z) \neq 0$ (Problem P6), $g_1(\frac{1+\xi}{2}z) \neq 0$ or $g_2(\frac{1-\xi}{2}z) \neq 0$ (Problem P6′). For characteristic functions (7), we have the next zero points of the function f:

$$z_k = (k-1)\pi, \quad k \in \mathbb{N}; \tag{$13_{1'}$}$$

$$z_k = (k-1/2)\pi, \quad k \in \mathbb{N}; \tag{$13_{1,2,3',4',5',6'}$}$$

$$z_k = k\pi, \quad k \in \mathbb{N}. \tag{$13_{3,4,5,6,2'}$}$$

Note that the zero points are the same for all ξ and they are on the vertical lines in the domain $D_{x,\xi} := \mathbb{R} \times D_\xi$. The point $x = 0$ is a zero point only for Problem P1′ and it is zero of the second order for all $\xi \in [0, 1]$.

The characteristic function (Problems P1–P5, P1′–P5′, $\xi \neq 0$) has a pole point $\tilde{p}$ if $g(\tilde{p}) = 0$ and $f(\tilde{p}/\xi) \neq 0$. For characteristic functions (7) we have the next zero points $\tilde{p}$ for the function g:

$$\tilde{p}_k = (k-1/2)\pi, \quad k \in \mathbb{N}; \tag{$14_{2,3,1',4'}$}$$

$$\tilde{p}_k = k\pi, \quad k \in \mathbb{N}; \tag{$14_{1,4,2',5'}$}$$

$$\tilde{p}_k = (k-1)\pi, \quad k \in \mathbb{N}; \tag{$14_{3'}$}$$

$$\tilde{p}_k = 2k\pi, \quad k \in \mathbb{N}. \tag{14_5}$$

In these cases the poles of the characteristic function are p_k and $\tilde{p}_k = \xi p_k$. So, the poles are on the hyperbolae $\xi x = \tilde{p}_k$, $k \in \mathbb{N}$, in the domain $D_{x,\xi}$. The point $x = 0$ is a pole point only for Problem P3′, and it is the pole of the second order for all $\xi \in (0, 1]$ and, in this case the hyperbola degenerates to the line $x = 0$. The characteristic function (Problems P2, P3, P1′, P4′) for $\xi = 0$ is an entire function, i.e., there are no points of poles.

Remark 4 (Constant eigenvalue points for Problems P1–P4, P1′–P5′)**.** *Note that $x = 0$ is not a constant eigenvalue point. All the positive zeroes and positive poles for these problems are of the first order. If we have $f(z_k) = g(p_l) = 0$ for some ξ, then this point $z_k = p_l = c$ is a constant eigenvalue point and $\gamma(c) \neq 0$. Geometrically we get constant eigenvalue points as the intersection vertical lines of zeroes and hyperbolae of poles in the domain $D_{x,\xi}$.*

Remark 5 (Constant eigenvalue points for Problem P5)**.** *We note that $x = 0$ is not a constant eigenvalue point. Positive zeroes are of the first order and positive poles for this problem are of the first or second order. If for some ξ we have $f(z_k) = g(p_l) = 0$, then this point $z_k = p_l = c$ is a constant eigenvalue point and $\gamma(c) \neq 0$. We have the* *first order poles at the points of constant eigenvalues. Geometrically we get constant eigenvalues points as the intersection of vertical lines of zeroes and hyperbolae of poles in the domain $D_{x,\xi}$, too.*

For Problem P6 $g(x) > 0$, $x \leqslant 0$ and for Problem P6′ $g_1(x) > 0$, $g_2(x) > 0$, $x \leqslant 0$. We have the next zero points $\tilde{p}$ for the function g, g_1, g_2 (see, Prop. 1):

$$\tilde{p}_m = m\pi, \quad m \in \mathbb{N}; \tag{$15_{6:g}$}$$

$$\tilde{p}_k = k\pi, \quad k \in \mathbb{N}; \tag{$15_{6':g_1}$}$$

$$\tilde{p}_l = (l - 1/2)\pi, \quad l \in \mathbb{N}. \tag{$15_{6':g_2}$}$$

For Problem P6 and Problem P6′ we have two families of poles p'_{m_1}, p''_{m_2} and p'_k, p''_l, where $\tilde{p}_{m_1} = \frac{1-\xi}{2}p'_{m_1}$, $\tilde{p}_{m_2} = \frac{1+\xi}{2}p''_{m_2}$ and $\tilde{p}_k = \frac{1-\xi}{2}p'_k$, $\tilde{p}_l = \frac{1+\xi}{2}p''_l$. These poles are on the hyperbolae $(1+\xi)x = 2\tilde{p}_m$, $(1-\xi)x = 2\tilde{p}_m$ (Problem P6) and hyperbolae $(1+\xi)x = 2\tilde{p}_k$, $(1-\xi)x = 2\tilde{p}_l$ (Problem P6′) in the domain $D_{x,\xi}$ (see, Fig. 2). In each case, the hyperbolae families are intersected only at the zero points of the function f, i.e., these intersection points are points of constant eigenvalues [17].

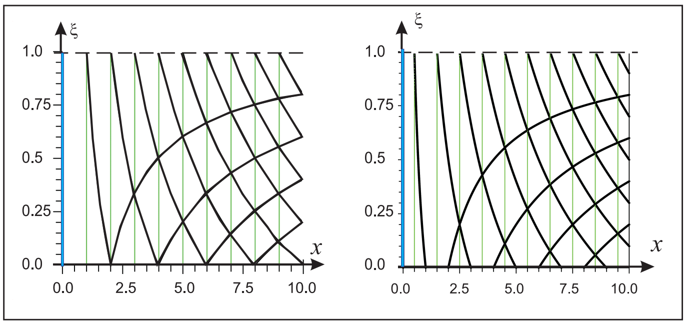


Fig. 2. Points of the first and second order poles of the real characteristic function in Case 6 (the left figure) and Case 6′ (the right figure).

Remark 6 (Constant eigenvalue points for Problems P6, P6′)**.** *All positive zeroes and positive poles for these problems are of the first order. Geometrically the three families of curves (zero lines, two families of pole hyperbolae) intersect together, i.e., if two families intersect, then their intersection point lies on the curve of the third family and this point is a constant eigenvalue point.*

Proposition 2. *All the constant eigenvalue points in Cases* 6 *and* 6′ *are on the hyperbolae* $x\xi = k\pi$, $k \in \mathbb{N}$ *and on x-axis* ($\xi = 0$)*, too.*

Proof. We get a constant eigenvalue point as the intersection of the two hyperbolae for $\xi > 0$ or (see, (15)):

$$\frac{1-\xi}{2}x = \tilde{p}_{m_1}, \qquad \frac{1+\xi}{2}x = \tilde{p}_{m_2}; \tag{16_6}$$

$$\frac{1-\xi}{2}x = \tilde{p}_k, \qquad \frac{1+\xi}{2}x = \tilde{p}_l. \tag{$16_{6'}$}$$

Note that $\tilde{p}_{m_1} < \tilde{p}_{m_2}$, $\tilde{p}_k < \tilde{p}_l$ for $\xi > 0$. If we add and subtract these equalities, then we get

$$x = \tilde{p}_{m_1} + \tilde{p}_{m_2}, \qquad x\xi = \tilde{p}_{m_2} - \tilde{p}_{m_1}; \tag{17_6}$$

$$x = \tilde{p}_k + \tilde{p}_l, \qquad x\xi = \tilde{p}_l - \tilde{p}_k. \tag{$17_{6'}$}$$

In the case $\xi = 0$, the proof follows from the first equalities (16). □

Let a line $\xi = const$ intersect the hyperbolae at the points h_k, where $h_k < h_{k+1}$, $k \in \mathbb{N}$. The points h_k, $k \in \mathbb{N}$ are poles (of the first or second order) or constant eigenvalue points for the real characteristic function. Let us define $h_0 = 0$. Then the real characteristic function γ is defined for $x \in \bar{P}_i := (h_{i-1}, h_i)$, $i \in \mathbb{N}$ and $\bar{P}_0 := (-\infty, 0)$. If h_i is a constant eigenvalue point c_j or c_0 and we have finite limits: $\lim_{x\to c_j} \gamma(x)$ or $\lim_{x\to c_0} \gamma(x)$, then we add this point to the interval, i.e., $\bar{P}_i := (h_{i-1}, c_j]$ or $\bar{P}_i := [c_{j-1}, h_i)$ or $\bar{P}_i := [c_{j-1}, c_j]$.

The spectrum of Sturm-Liouville problems (1)–(3) and (1), (4), (3) were investigated in [8, 9, 11, 17]. Lemmas on the existence zeroes, poles, minimums and maximums of the characteristic functions and conditions on the existence of constant eigenvalues are presented there. We note (see, [11]) that two negative real eigenvalues can exist in the negative part of the real spectrum in problems P3 and P3′ for some γ and ξ values. Negative multiple and complex eigenvalues can exist as well. In other cases of nonlocal boundary conditions, there exists one negative real eigenvalue for particular values of the parameter γ.

4 Critical points of real characteristic function

The point x_{cr} is a *critical point* of the real characteristic function, if $\gamma'(x_{cr}) = 0$. Critical points of the characteristic function are important for the investigation of multiple eigenvalues. Critical points of the characteristic function are maximum and minimum points of this function (see, Fig. 1). Generalized eigenfunctions exist for these points [9]. The generalized eigenfunctions exist at constant eigenvalue points, too. If this point is critical point, then we have generalized eigenfunctions of the second order, else generalized eigenfunctions are of the first order.

Now, note that $p_k = p_k(\xi)$, $c_j = c_j(\xi)$, $x_{cr} = x_{cr}(\xi)$, and recall the case with non fixed ξ. For any critical point $x_{cr}(\xi)$, we define the *critical point* $(x_{cr}(\xi), \xi) \in D_{x,\xi} := \mathbb{R} \times D_\xi \subset \mathbb{R}_x \times \mathbb{R}_\xi$. If the critical point is an extremum, i.e. the maximum or minimum point, then we use the notation “extremum” instead of “critical”. Note that the property “to be a critical point” or “to be extremum” in the $\mathbb{R}_{x,\xi}$ is only in the x-direction. The critical points $(x_{cr}(\xi), \xi)$ depend on the parameter ξ continuously. The set of these points is curves in the domain $\mathbb{R}_{x,\xi}$.

4.1 Investigation of the few auxiliary functions

Let us consider functions [17] (see, Fig. 3 and Fig. 4):

$$\varphi_-(x) := x\coth x; \qquad \varphi_+(x) := x\cot x;$$
$$\psi_-(x) := x\tanh x; \qquad \psi_+(x) := -x\tan x$$

for $x \in \mathbb{R}$ and the functions:

$$\varphi(x) := \begin{cases} \varphi_-(x) & \text{for } x \leqslant 0, \\ \varphi_+(x) & \text{for } x \geqslant 0; \end{cases} \qquad \psi(x) := \begin{cases} \psi_-(x) & \text{for } x \leqslant 0, \\ \psi_+(x) & \text{for } x \geqslant 0. \end{cases}$$

These two functions are related by the equality

$$\varphi(x)\cdot\psi(x) = -x|x|. \tag{18}$$

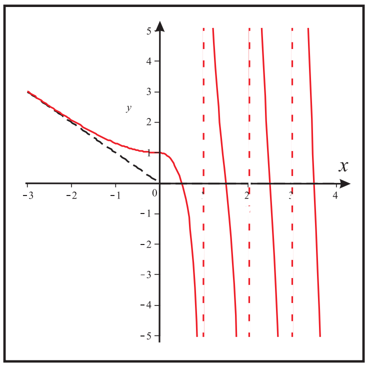


Fig. 3. The function $\varphi(x)$.

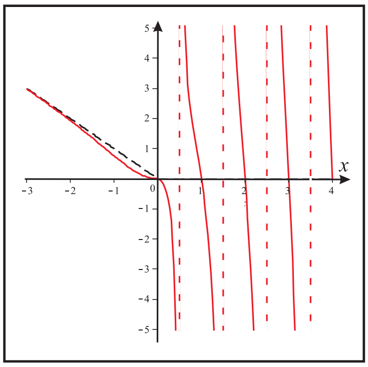


Fig. 4. The function $\psi(x)$.

Proposition 3. *Functions φ, ψ are positive decreasing functions for $x < 0$, φ is a decreasing function in the intervals $[0,\pi)$, $(\pi k, \pi(k+1))$, $k \in \mathbb{N}$ and ψ is a decreasing function in the intervals $[0,\pi/2)$, $(\pi(k-1/2), \pi(k+1/2))$, $k \in \mathbb{N}$.*

Proof. The derivatives of this function are equal to:

$$\varphi'_-(x) := \frac{\sinh(2x)-2x}{\cosh(2x)-1}; \qquad \varphi'_+(x) := -\frac{\sin(2x)-2x}{\cos(2x)-1};$$
$$\psi'_-(x) := \frac{\sinh(2x)+2x}{\cosh(2x)+1}; \qquad \psi'_+(x) := -\frac{\sin(2x)+2x}{\cos(2x)+1}.$$

So, the derivatives of the function φ and ψ are negative for $x \neq 0$. The positiveness of the functions φ, ψ for $x < 0$ is evident. □

Corollary 2. *The properties $\lim_{x\to 0}\varphi(x) = \varphi(0) = 1$, $\psi(0) = 0$, $\lim_{x\to 0}\varphi'(x) = \varphi'(0) = 0$, $\psi'(0) = 0$ are valid.*

Corollary 3. *The inequality $x\coth(x) \geqslant 1$ is valid for all $x \in \mathbb{R}$ and the equality is true only for $x = 0$.*

Proposition 4. *The inequality*

$$\coth x - \tanh x < \frac{x}{\sinh^2 x} \tag{19}$$

is valid for $x > 0$.

Proof. We derive for $x > 0$:

$$\coth x - \tanh x = \frac{1}{\sinh x \cosh x} < \frac{x}{\sinh x \cosh x}\coth x = \frac{x}{\sinh^2 x}.$$

□

Proposition 5. *The function $h_0(x) := x(\sinh x + x)/(\cosh x + 1)$ is an increasing positive function for $x \in (0, +\infty)$ and $h_0(0) = 0$.*

Proof. The derivative of the function

$$S_0(x) := \sinh x + \sinh x \cosh x - x^2 \sinh x + 3x + 3x\cosh x$$

is

$$S_0'(x) = 4\cosh x + \cosh x(\cosh x - x^2) + \sinh^2 x + x\sinh x + 3 > 0.$$

Since $S_0(0) = 0$, we get that $S_0(x) > 0$ for $x > 0$. The derivative of the function h_0 is

$$h_0'(x) := \frac{\sinh x + \sinh x\cosh x - x^2 \sinh x + 3x + 3x\cosh x}{(\cosh x + 1)^2} = \frac{S_0(x)}{(\cosh x + 1)^2}.$$

Thus, $h_0'(x) > 0$ for $x > 0$. The positiveness of the function h_0 and condition $h_0(0) = 0$ is evident. □

Proposition 6. *The function $h_1(x) := (12 + 6x\sinh x - 12\cosh x)/(\cosh x - 1)/x^2$ is a decreasing positive function for $x \in (0, +\infty)$ and $h_1(0) = 1$, $h_1(+\infty) = 0$.*

Proof. The derivatives of the function

$$S_1(x) := -4\cosh x + x^2 + x\sinh x + 4$$

are

$$S_1'(x) = -3\sinh x + 2x + x\cosh x, \qquad S_1''(x) = -2\cosh x + 2 + x\sinh x,$$

$$S_1'''(x) = -\sinh x + x\cosh x, \qquad S_1^{(4)}(x) = x\sinh x > 0.$$

Since $S_1(0) = S_1'(0) = S_1''(0) = S_1'''(0) = 0$, we get that $S_1(x) > 0$ for $x > 0$. The derivative for the function h_1 is

$$h_1'(x) := \frac{24\cosh x - 6x^2 - 6x\sinh x - 24}{(\cosh x - 1)x^3} = -\frac{6S_1(x)}{(\cosh x - 1)x^3}.$$

So, $h_1'(x) < 0$ for $x > 0$. The positiveness of the function h_1 and conditions $h_1(0) = 1$, $h_1(+\infty) = 0$ are evident. □

Proposition 7. *The function $h_2(x) := x^3/(\sinh x - x)/6$ is a decreasing positive function for $x \in (0, +\infty)$ and $h_2(0) = 1$, $h_2(+\infty) = 0$.*

Proof. The derivatives of the function

$$S_2(x) := -3\sinh x + 2x + x\cosh x$$

are

$$\begin{aligned} S_2'(x) &= -2\cosh x + 2 + x\sinh x, \qquad S_2''(x) = -\sinh x + x\cosh x, \\ S_2'''(x) &= x\sinh x > 0. \end{aligned}$$

Since $S_2(0) = S_2'(0) = S_2''(0) = 0$, we get that $S_2(x) > 0$ for $x > 0$. The derivative for the function h_2 is

$$h_2'(x) := \frac{x^2}{6} \cdot \frac{3\sinh x - 2x - x\cosh x}{(\sinh x - x)^2} = -\frac{x^2 S_2(x)}{6(\sinh x - x)^2}.$$

Consequently, $h_2'(x) < 0$ for $x > 0$. The positiveness of the function h_2 and conditions $h_2(0) = 1$, $h_2(+\infty) = 0$ are evident. □

Let us consider two functions for $x > 0$:

$$\begin{aligned} V(x) &:= 4\cosh^3 x + 4x\sinh x\cosh x - 4\cosh x - 4x^2 - 4x\sinh x, \\ S(x) &:= -8\cosh^3 x - 8\cosh^2 x + 4x\sinh x\cosh^2 x + 4x^2\cosh x \\ &\quad + 4x\sinh x\cosh x + 8\cosh x + 8 + 4x^2. \end{aligned}$$

We find derivatives of the first function:

$$\begin{aligned} V'(x) &= 12\cosh^2 x\sinh x + 4x\cosh^2 x + 4\sinh x\cosh x + 4x\sinh^2 x \\ &\quad - 8\sinh x - 8x - 4x\cosh x, \\ V''(x) &= 24\cosh x\sinh^2 x + 12(\cosh^2 x - 1)\cosh x + 4x\sinh x(4\cosh x - 1) \\ &\quad + 8(\cosh^2 x - 1) + 8\sinh^2 x > 0. \end{aligned}$$

Since $V(0) = V'(0) = 0$, we get that $V(x) > 0$ for $x > 0$.

The derivatives of the second function are:

$$\begin{aligned} S'(x) &= -20\cosh^2 x\sinh x - 12\sinh x\cosh x + 4x\cosh^3 x \\ &\quad + 8x\sinh^2 x\cosh x + 8x\cosh x + 4x^2\sinh x + 4x\cosh^2 x \\ &\quad + 4x\sinh^2 x + 8\sinh x + 8x, \\ S''(x) &= -32\cosh x\sinh^2 x - 16\cosh^3 x - 8\cosh^2 x - 8\sinh^2 x \\ &\quad + 28x\sinh x\cosh^2 x + 8x\sinh^3 x + 16x\sinh x \\ &\quad + 16\cosh x + 4x^2\cosh x + 16x\sinh x\cosh x + 8, \end{aligned}$$

$$
\begin{aligned}
S'''(x) =& -24\sinh^3 x - 84\cosh^2 x\sinh x - 16\sinh x\cosh x + 28x\cosh^3 x\\
&+ 80x\sinh^2 x\cosh x + 24x\cosh x + 32\sinh x + 4x^2\sinh x\\
&+ 16x\cosh^2 x + 16x\sinh^2 x,\\
S^{(4)}(x) =& -160\cosh x\sinh^2 x - 56\cosh^3 x + 244x\sinh x\cosh^2 x\\
&+ 80x\sinh^3 x + 32x\sinh x + 56\cosh x + 4x^2\cosh x\\
&+ 64x\sinh x\cosh x,\\
S^{(5)}(x) =& -80\sinh^3 x - 244\cosh^2 x\sinh x + 244x\cosh^3 x\\
&+ 728x\sinh^2 x\cosh x + 40x\cosh x + 88\sinh x + 4x^2\sinh x\\
&+ 64x\cosh^2 x + 64\sinh x\cosh x + 64x\sinh^2 x,\\
S^{(6)}(x) =& 2188x\sinh x\cosh^2 x + 728x\sinh^3 x + 48x\sinh x + 128\cosh x\\
&+ 4x^2\cosh x + 256x\sinh x\cosh x + 128\cosh^2 x + 128\sinh^2 x > 0.
\end{aligned}
$$

Since $S(0) = S'(0) = S''(0) = S^{(3)}(0) = S^{(4)}(0) = S^{(5)}(0) = 0$, we obtain that $S(x) > 0$ for $x > 0$.

If we define the functions:

$$
\begin{aligned}
G_0(x) &:= x\frac{(\cosh x+1)^2 - (\sinh x + x)\sinh x}{(\cosh x+1)(\sinh x+x)} = x\frac{2+2\cosh x - x\sinh x}{(\cosh x+1)(\sinh x+x)},\\
F_0(x) &:= x\frac{(\cosh x-1)^2 - (\sinh x - x)\sinh x}{(\cosh x-1)(\sinh x-x)} = x\frac{2+x\sinh x - 2\cosh x}{(\cosh x-1)(\sinh x-x)},
\end{aligned}
$$

then

$$
F_0(x) - G_0(x/2) = \frac{x}{2}\frac{S(x/2)}{V(x/2)\sinh(x/2)} > 0, \quad x > 0. \tag{20}
$$

Corollary 4 (see, Proposition 6 and Proposition 7)**.** *The function $F_0(x) = h_1(x)h_2(x)$ is a decreasing positive function for $x \in (0,+\infty)$ and $F_0(0) = 1$, $F_0(+\infty) = 0$.*

Let us define positive functions for $x > 0$:

$$
F_1(x) := \frac{\sinh x - x}{\cosh x - 1} = \varphi'_-(x/2), \qquad G_1(x) := \frac{\sinh x + x}{\cosh x + 1} = \psi'_-(x/2).
$$

Proposition 8. *The function $F_1(x)$ is an increasing positive function for $x \in (0,+\infty)$ and $F_1(0) = 0$.*

Proof. We derive for $x > 0$:

$$
x\coth x > 1 \;\Rightarrow\; 2\sinh x < \sinh x + x\cosh x.
$$

Now we integrate the latter inequality from 0 to x and get

$$
\begin{aligned}
&2\cosh x - 2 < x\sinh x \;\Rightarrow\; \sinh^2 x - x\sinh x < \cosh^2 x - 2\cosh x + 1\\
&\Rightarrow\; (\sinh x - x)\sinh x < (\cosh x - 1)^2 \;\Rightarrow\; \frac{(\sinh x - x)\sinh x}{(\cosh x-1)^2} < 1.
\end{aligned}
$$

As a result, we have

$$0 < F_1'(x) = 1 - \frac{(\sinh x - x)\sinh x}{(\cosh x - 1)^2} < 1. \tag{21}$$

The positiveness of the function F_1 and condition $F_1(0) = 0$ are evident. □

Corollary 5. *The inequalities* $0 < \varphi_-''(x) < 1$ *are valid for* $x < 0$.

Corollary 6. *The function* $h_3(x) := x(\sinh x - x)/(\cosh x - 1)/2$ *is an increasing positive function for* $x \in (0, +\infty)$ *and* $h_3(0) = 0$.

Remark 7. *The function* $G_1(x)$ *is positive and* $G_1(x) > \tanh x$.

Functions F_1 and G_1 are both positive. Then we have the positive function

$$H_1(x;y) := \frac{F_1(x)}{G_1(yx)} = \frac{\cosh(yx)+1}{\sinh(yx)+yx} \cdot \frac{\sinh x - x}{\cosh x - 1} = \frac{\varphi_-'(x/2)}{\psi_-'(yx/2)}$$

for all $y > 0$. The derivative of this function is

$$H_1'(x;y) := \frac{F_1(x)}{G_1(yx)x}\left(x\frac{F_1'(x)}{F_1(x)} - yx\frac{G_1'(yx)}{G_1(yx)}\right) = \frac{F_1(x)}{G_1(yx)x}\big(F_0(x) - G_0(yx)\big).$$

The graphs of the functions $F_0(x)$ and $G_0(yx)$ for $y \geqslant 1/2$ are presented in Fig. 5.

Corollary 7. *If the parameter* $y \geqslant 1/2$*, then the function* $H_1(x;y)$ *is an increasing function for* $x > 0$.

Remark 8. *The value* $y = 1/2$ *is not the limit value. From the Taylor series*

$$F_0(x) - G_0(yx) = \big(-1/15 + (1/3)y^2\big)x^2 + \big(919/403200 - (2/45)y^4\big)x^4 + \mathcal{O}\big(x^6\big)$$

it follows that this value is $y = \sqrt{5}/5 \approx 0.447$.

Let us consider a function $H(x;y)$ for $x > 0$

$$\begin{aligned} H(x;y) :&= \frac{\coth x - x/\sinh^2 x}{y\tanh(yx) + y^2x/\cosh^2(yx)} = \frac{\cosh(2yx)+1}{y(\sinh(2yx)+2yx)} \cdot \frac{\sinh(2x)-2x}{\cosh(2x)-1} \\ &= \frac{1}{y}H_1(2x;y) = \frac{1}{h_0(2yx)} \cdot 2xF_1(2x). \end{aligned}$$

Corollary 8 (see, Proposition 5)**.** *For fixed* $x > 0$*, the function* $H_x(y) := H(x;y),\ \ y > 0$ *is a decreasing function.*

Corollary 9. *For fixed* $y \geqslant 1/2$ *the function* $H_y(x) := H(x;y)$ *is an increasing function for all* $x > 0$.

The graphs of the functions $H(x;y)$ for some $y > 0$ are presented in Fig. 6. We note that for $y > 0$

$$\lim_{x\to 0} H(x;y) = \frac{1}{3y^2}, \tag{22}$$

$$\lim_{x\to +\infty} H(x;y) = \frac{1}{y}. \tag{23}$$

Finally, we consider the functions

$$G(x;y) := x\coth x - yx\tanh(yx) = \varphi_-(x) - \psi_-(yx), \quad x\in\mathbb{R}, \tag{24}$$

$$\tilde{G}(x;y) := yx\coth(yx) - x\tanh(x) = \varphi_-(yx) - \psi_-(x), \quad x\in\mathbb{R} \tag{25}$$

for $y \geqslant 0$ (see, Fig. 7 and Fig. 9). Since these functions are even, we investigate them for $x > 0$.

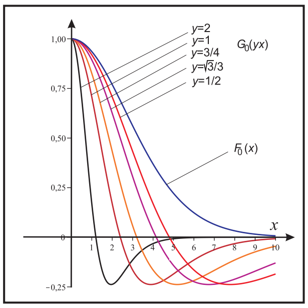


Fig. 5. The functions $F_0(x)$ and $G_0(yx)$.

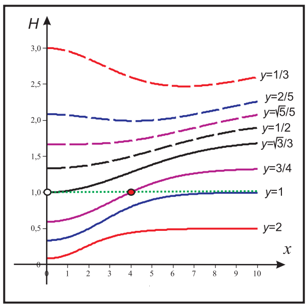


Fig. 6. The function $H(x;y)$.

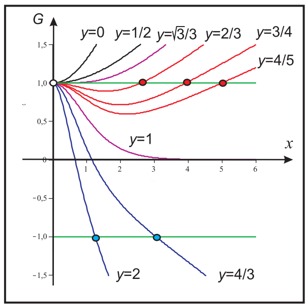


Fig. 7. The function $G(x;y)$, $x > 0$.

Let us begin with function G. We note that $G(0;y) = 1$. The derivative of this function

$$G'(x;y) = \coth x - x/\sinh^2 x - y\tanh(yx) - y^2x/\cosh^2(yx)$$

and

$$G'(x;y) = \begin{cases} yG_1(yx)\big(H(x;y)-1\big) & \text{for } \ y>0, \\ F_1(2x) & \text{for } \ y=0. \end{cases} \tag{26}$$

Lemma 1. *The function $G(x;y)$ is even. It has the following properties:*

(i) *If $y \in [0, \sqrt{3}/3]$, then G is an increasing function for $x \geqslant 0$ and $x = 0$ is the minimum point of the function G;*

(ii) *If $y \in (\sqrt{3}/3, 1)$, then there exists $x_{min}(y) > 0$ such that the function G is a decreasing function for $0 < x < x_{min}(y)$, and G is an increasing function for*

$x > x_{min}(y)$. There exists $x_{min}(y) > 0$, $0 < G(x_{min}(y); y)) < 1$ which is the minimum point of function G and $x = 0$ is the maximum point of function G ($-x_{min}(y)$ is the minimum point, too);

(iii) *If $y \in [1, +\infty)$, then G is a decreasing function for $x \geqslant 0$ and $x = 0$ is the maximum point of function G;*

(iv) *For fixed $x > 0$, the function $G_x(y) := G(x; y)$ is a decreasing function.*

For all $y \geqslant 0$, we have $G(0; y) = 1$ and

$$\lim_{x\to+\infty} G(x; y)/x = 1 - y, \quad \lim_{x\to-\infty} G(x; y)/x = -(1 - y), \quad \lim_{x\to\pm\infty} G(x; 1) = 0.$$

Proof. $G'(x; y) > 0 (= 0, < 0)$ if and only if $H(x; y) > 1 (= 1, < 1)$. If $y \in (\sqrt{3}/3, 1)$, then $H(0, y) < 1 < H(+\infty; y)$ (see, limits (22) and (23)). As H is an increasing function, we have only one point $x_{min}(y)$ such that $H(x_{min}(y); y) = 1$.

If $y \in [1, +\infty)$, then $H(x; y) < 1$ (in this case $H(+\infty; y) < 1$), and if $y \in [1/2, \sqrt{3}/3)$, then $H(x; y) > 1$ (in this case $H(+0; y) > 1$). If $y = \sqrt{3}/3$ then $G(0; \sqrt{3}/3) = 1$, but $G(x; \sqrt{3}/3) > 1$ for $x > 0$.

For $y \in (0, 1/2)$, we have (see, Corollary 8)

$$H(x; y) > H(x; 1/2) > 1.$$

So, G is an increasing function for such y.

The derivative $G'_x(y) = \frac{\partial}{\partial y} G(x; y) = -x\psi'_-(yx) < 0$ (see, Proposition 3).

The other properties of the function G are evident. □

Corollary 10. *If $x \in (-\infty, 0)$, then*

$$0 < \psi(x) < \varphi(x) < \psi(x) + 1. \tag{27}$$

Proof. Let us consider function $\tilde{G}$. We note that $\tilde{G}(0; y) = 1$,

$$\tilde{G}(x; y) = \begin{cases} G(yx; 1/y) & \text{for } y > 0, \\ 1 - \psi(x) & \text{for } y = 0. \end{cases} \tag{28}$$

The derivatives are equal to

$$\tilde{G}'_x(y) := \frac{\partial}{\partial y}\tilde{G}(x; y) = x\varphi'_-(yx) > 0, \tag{29}$$

$$\tilde{G}'_y(x) := \frac{\partial}{\partial x}\tilde{G}(x; y) = G_1(x)\big(\tilde{H}(x; y) - 1\big), \tag{30}$$

where

$$\tilde{H}(x; y) = \begin{cases} H(yx; 1/y) & \text{for } y > 0, \\ 0 & \text{for } y = 0. \end{cases} \tag{31}$$

It follows that

$$\tilde{H}(x;y) = \frac{2yxF_1(2yx)}{h_0(2x)} = yH_1(2yx;1/y). \tag{32}$$

So, we have the following properties for function $\tilde{H}$.

Corollary 11. *For fixed $x > 0$, the function $\tilde{H}_x(y) := \tilde{H}(x;y)$, $y > 0$ is an increasing function.*

Corollary 12. *For fixed $0 < y \leqslant 2$, the function $\tilde{H}_y(x) := \tilde{H}(x;y)$ is an increasing function for all $x > 0$.*

The graphs of the functions $\tilde{H}(x;y)$ for some $y > 0$ are presented in Fig. 8.
We see that for $y > 0$

$$\lim_{x\to 0} \tilde{H}(x;y) = \frac{y^2}{3}, \tag{33}$$

$$\lim_{x\to +\infty} \tilde{H}(x;y) = y. \tag{34}$$

From these properties we derive the following lemma (the proof is the same as that of Lemma 1).

Lemma 2. *The function $\tilde{G}(x;y)$ is even. It has the following properties:*

(i) *If $y \in [0,1]$, then $\tilde{G}$ is a decreasing function for $x \geqslant 0$ and $x = 0$ is the maximum point of function $\tilde{G}$;*

(ii) *If $y \in (1, \sqrt{3})$, then there exists $\tilde{x}_{min}(y) > 0$ such that function $\tilde{G}$ is a decreasing function for $0 < x < \tilde{x}_{min}(y)$, and $\tilde{G}$ is an increasing function for $x > \tilde{x}_{min}(y)$. There exists $\tilde{x}_{min}(y) > 0$, $0 < \tilde{G}(\tilde{x}_{min}(y);y)) < 1$ which is the minimum point of function G and $x = 0$ is the maximum point of function $\tilde{G}$ $(-\tilde{x}_{min}(y)$ is the minimum point, too);*

(iii) *If $y \in [\sqrt{3}, +\infty)$, then $\tilde{G}$ is an increasing function for $x \geqslant 0$ and $x = 0$ is the minimum point of the function $\tilde{G}$;*

(iv) *For fixed $x > 0$, the function $\tilde{G}_x(y) := \tilde{G}(x;y)$ is an increasing function.*

For all $y \geqslant 0$, we have $\tilde{G}(0;y) = 1$ and

$$\lim_{x\to+\infty} \tilde{G}(x;y)/x = y-1, \quad \lim_{x\to-\infty} \tilde{G}(x;y)/x = -(y-1), \quad \lim_{x\to\pm\infty} \tilde{G}(x;1) = 0.$$

Lemma 3. *The functions φ and ψ satisfy the Riccati differential equation*

$$y'(x) = \frac{1}{x}y(x) - \frac{1}{x}y^2(x) - |x|; \tag{35}$$

Proof. We prove the lemma by substituting the functions φ and ψ directly into differential equations. □

4.2 Critical points equation and its properties

Lemma 4. *There are no critical points in Case 6 and Case 6′.*

Proof. The functions

$$\gamma_6(x) := \frac{1}{1-\xi}\varphi\bigl(\tfrac{1-\xi}{2}x\bigr) + \frac{1}{1+\xi}\varphi\bigl(\tfrac{1+\xi}{2}x\bigr),$$
$$\gamma_{6'}(x) := \frac{1}{1-\xi}\varphi\bigl(\tfrac{1-\xi}{2}x\bigr) + \frac{1}{1+\xi}\psi\bigl(\tfrac{1+\xi}{2}x\bigr)$$

are decreasing functions as the sum of such functions. □

Let us consider the characteristic functions for Problems P1–P5, P1′–P5′. These functions are of the form (see, (7))

$$\gamma(x;\xi) = \varkappa(\xi)\frac{f(x)}{g(\xi x)} \tag{36}$$

where $\varkappa(\xi) = \frac{1}{\xi}, 1, \frac{2}{\xi^2}, \xi$. The derivative (by x) of this function

$$\gamma'(x;\xi) = \varkappa(\xi)\frac{f'(x)g(\xi x) - f(x)g'(\xi x)\xi}{g^2(\xi x)}. \tag{37}$$

We can write the condition on the critical point ($\gamma' = 0$) in the interval (h_{i-1}, h_i) as

$$f'(x) = \xi f(x)\frac{g'(\xi x)}{g(\xi x)}, \tag{38}$$

because $g(\xi x) \neq 0$ for $x \in (h_{i-1}, h_i)$. If x_c is a critical point and $f(x_c) = 0$ then from (38) we derive $f'(x_c) = 0$, i.e., the critical point is zero of the second order for the function f, however, Proposition 1 declares that there are zeroes only of the first order for $x \neq 0$. Thus, $f(x_c) \neq 0$, and we can write equality (38) as follows:

$$F(x,\xi) := D_{ln}f(x) - D_{ln}g(\xi x) = 0, \tag{39}$$

where

$$D_{ln}f(x) := x\log'|f(x)| = x\frac{f'(x)}{f(x)} \quad \text{for } f(x) \neq 0. \tag{40}$$

Consequently, we can rewrite equality (37) as

$$\gamma'(x;\xi) = \frac{1}{x}\varkappa(\xi)\gamma(x;\xi)F(x;\xi) = \frac{1}{x}\varkappa(\xi)\gamma(x;\xi)\bigl(D_{ln}f(x) - D_{ln}g(\xi x)\bigr). \tag{41}$$

Proposition 9. *For D_{ln} the next properties are valid:*

$$D_{ln}(fg) = D_{ln}f + D_{ln}g;\ D_{ln}(f^g) = gD_{ln}f + gD_{ln}g\log f, \quad f > 0;$$
$$D_{ln}f^\alpha = \alpha D_{ln}f, \quad \alpha \in \mathbb{R}; D_{ln}(x^\alpha f) = D_{ln}f + \alpha, \quad \alpha \in \mathbb{R};$$
$$D_{ln}(x^\alpha) = \alpha, \quad \alpha \in \mathbb{R}; \quad D_{ln}\bigl(f \circ g(x)\bigr) = (D_{ln}f)\bigl(g(x)\bigr) \cdot D_{ln}g(x);$$
$$D_{ln}\bigl(f(\alpha x)\bigr) = (D_{ln}f)(\alpha x), \quad \alpha \in \mathbb{R}.$$

So, we obtain

$D_{ln}\sin x = x\cot x = \varphi_+(x);$
$D_{ln}\frac{\cosh x-1}{x^2/2} = x\coth(x/2) - 2 = 2\big(\varphi_-(x/2) - 1\big);$
$D_{ln}\frac{\sin x}{x} = x\cot x - 1 = \varphi_+(x) - 1;$
$D_{ln}\frac{\sinh x}{x} = x\coth x - 1 = \varphi_-(x) - 1;$
$D_{ln}\sinh x = x\coth x = \varphi_-(x);$
$D_{ln}(x\sinh x) = x\coth x + 1 = \varphi_-(x) + 1;$
$D_{ln}\cos x = -x\tan x = \psi_+(x);$
$D_{ln}\frac{1-\cos x}{x^2/2} = x\cot(x/2) - 2 = 2\big(\varphi_+(x/2) - 1\big);$
$D_{ln}\cosh x = x\tanh x = \psi_-(x);$
$D_{ln}(-x\sin x) = x\cot x + 1 = \varphi_+(x) + 1;$

and equality (39) is valid with:

$$D_{ln}f(x) = \psi(x), \qquad D_{ln}g(x) = \varphi(x) - 1; \tag{$42_{1,5'}$}$$
$$D_{ln}f(x) = \psi(x), \qquad D_{ln}g(x) = \psi(x); \tag{$42_{2,4'}$}$$
$$D_{ln}f(x) = \varphi(x) - 1, \qquad D_{ln}g(x) = \psi(x); \tag{42_3}$$
$$D_{ln}f(x) = \varphi(x) - 1, \qquad D_{ln}g(x) = \varphi(x) - 1; \tag{$42_{4,2'}$}$$
$$D_{ln}f(x) = \varphi(x) - 1, \qquad D_{ln}g(x) = 2\big(\varphi(x/2) - 1\big); \tag{42_5}$$
$$D_{ln}f(x) = \varphi(x) + 1, \qquad D_{ln}g(x) = \psi(x); \tag{$42_{1'}$}$$
$$D_{ln}f(x) = \psi(x), \qquad D_{ln}g(x) = \varphi(x) + 1. \tag{$42_{3'}$}$$

Afterwards we derive the next expressions for function F:

$$F(x;\xi) = \psi(x) - \varphi(\xi x) + 1; \tag{$43_{1,5'}$}$$
$$F(x;\xi) = \psi(x) - \psi(\xi x); \tag{$43_{2,4'}$}$$
$$F(x;\xi) = \varphi(x) - \psi(\xi x) - 1; \tag{43_3}$$
$$F(x;\xi) = \varphi(x) - \varphi(\xi x); \tag{$43_{4,2'}$}$$
$$F(x;\xi) = \varphi(x) - 2\varphi(\xi x/2) + 1; \tag{43_5}$$
$$F(x;\xi) = \varphi(x) - \psi(\xi x) + 1; \tag{$43_{1'}$}$$
$$F(x;\xi) = \psi(x) - \varphi(\xi x) - 1. \tag{$43_{3'}$}$$

The equation $F(x;\xi) = 0$ defines a set of critical points in the domain $D_{x,\xi}$. This set is graphs of functions $\xi = \xi_{cr}(x)$ locally, as in all the cases $\frac{\partial F}{\partial \xi}\mathrm{sign}(x) > 0$ ($\frac{\partial F}{\partial \xi} = -x\varphi'(\xi x)$ or $\frac{\partial F}{\partial \xi} = -x\psi'(\xi x)$ or $\frac{\partial F}{\partial \xi} = -x\varphi'(\xi x/2)$) for $x \neq 0$.

Remark 9 (The case $\xi = 0$)**.** *For* $\xi = 0$, *(Problems* P2, P3, P1′, P4′*)* $\varphi(0) = 1$, $\psi(0) = 0$, *but in this case* $\frac{\partial F}{\partial x} \neq 0$. *So, we get the critical points as roots* z *of the equations:*

$\psi(x) = 0$ *(Problem* $2, 4'$*),* $\varphi(x) = 1$ *(Problem* 3*),* $\varphi(x) = -1$ *(Problem* $1'$*). At the points* $(z,0) \in D_{x,\xi}$*,* $z \neq 0$ *the curve* $F(x,\xi)$ *has a vertical tangent line. If* $0 \notin D_\xi$*, then* $\lim_{x\to z} \xi(x) = +\infty$ *where* z *is a nonzero root of the equation:* $\psi(x) = 0$ *(Problems* $1, 5'$*),* $\varphi(x) = 1$ *(Problems* $4, 2'$*),* $\varphi(x) = 1$ *(Problem* 5*),* $\psi(x) = 0$ *(Problem* $3'$*).*

In the domain $D_{x,\xi}$ there are two families of curves: vertical lines of the points of zeroes, and hyperbolae of the points of poles. The intersection points of these two families are constant eigenvalue points. We denote by D^c the complement of the union of these two families and use the notation $D^c_- := \{(x,\xi) \in D^c\colon x < 0\}$, $D^c_+ := \{(x,\xi) \in D^c\colon x > 0\}$. This complement D^c_+ is an infinite union of curvilinear subfigures: triangles, and quadrangles, and so on (see, Fig. 10). Then either the left or right side of these figures is vertical lines, and the other sides are hyperbolae, or $\xi = 0, 1$. The curves $F(x,\xi) = 0$ ($\gamma' = 0$) are inside these subdomains, and have no common points with the vertical sides ($\gamma = 0$) or parts of hyperbolae ($\gamma = \infty$), except the vertices (constant eigenvalue points) or lines $\xi = 0, 1$.

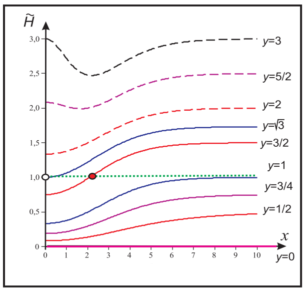


Fig. 8. The function $\tilde{H}(x;y)$.

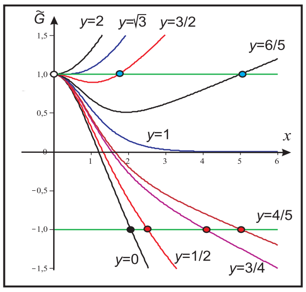


Fig. 9. The function $\tilde{G}(x;y)$, $x > 0$.

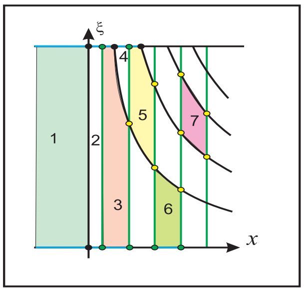


Fig. 10. The subdomains of the domain D^c.

From (42) and Lemma 3 it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} f(x) = \frac{1}{x} D_{ln} f(x) - \frac{1}{x}\big(D_{ln} f(x)\big)^2 - |x|, \qquad (44_{1,2,3',4',5'})$$

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} f(x) = -\frac{1}{x} D_{ln} f(x) - \frac{1}{x}\big(D_{ln} f(x)\big)^2 - |x|, \qquad (44_{3,4,5,2'})$$

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} f(x) = \frac{3}{x} D_{ln} f(x) - \frac{1}{x}\big(D_{ln} f(x)\big)^2 - \frac{2}{x} - |x|; \qquad (44_{1'})$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} g(x) = -\frac{1}{x} D_{ln} g(x) - \frac{1}{x}\big(D_{ln} g(x)\big)^2 - |x|, \qquad (45_{1,4,2',5'})$$

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} g(x) = \frac{1}{x} D_{ln} g(x) - \frac{1}{x}\big(D_{ln} g(x)\big)^2 - |x|, \qquad (45_{2,3,1',4'})$$

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} g(x) = -\frac{1}{x} D_{ln} g(x) - \frac{1}{2x}\big(D_{ln} g(x)\big)^2 - \frac{1}{2}|x|, \qquad (45_{5})$$

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} g(x) = \frac{3}{x} D_{ln} g(x) - \frac{1}{x}\big(D_{ln} g(x)\big)^2 - \frac{2}{x} - |x|. \qquad (45_{3'})$$

We rewrite the latter formulae at the point ξx:

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = -\frac{1}{x} D_{ln} g(\xi x) - \frac{1}{x}\big(D_{ln} g(\xi x)\big)^2 - \xi^2 |x|, \tag{$46_{1,4,2',5'}$}$$

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = \frac{1}{x} D_{ln} g(\xi x) - \frac{1}{x}\big(D_{ln} g(\xi x)\big)^2 - \xi^2 |x|, \tag{$46_{2,3,1',4'}$}$$

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = -\frac{1}{x} D_{ln} g(\xi x) - \frac{1}{2x}\big(D_{ln} g(\xi x)\big)^2 - \frac{\xi^2}{2} |x|, \tag{46_5}$$

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = \frac{3}{x} D_{ln} g(\xi x) - \frac{1}{x}\big(D_{ln} g(\xi x)\big)^2 - \frac{2}{x} - \xi^2 |x|. \tag{$46_{3'}$}$$

At the critical point $D_{ln} g(\xi x) = D_{ln} f(x)$:

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = -\frac{1}{x} D_{ln} f(x) - \frac{1}{x}\big(D_{ln} f(x)\big)^2 - \xi^2 |x|, \tag{$47_{1,4,2',5'}$}$$

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = \frac{1}{x} D_{ln} f(x) - \frac{1}{x}\big(D_{ln} f(x)\big)^2 - \xi^2 |x|, \tag{$47_{2,3,1',4'}$}$$

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = -\frac{1}{x} D_{ln} f(x) - \frac{1}{2x}\big(D_{ln} f(x)\big)^2 - \frac{\xi^2}{2} |x|, \tag{47_5}$$

$$\frac{\partial}{\partial x} D_{ln} g(\xi x) = \frac{3}{x} D_{ln} f(x) - \frac{1}{x}\big(D_{ln} f(x)\big)^2 - \frac{2}{x} - \xi^2 |x|. \tag{$47_{3'}$}$$

Finally, we get the expressions for the derivatives:

$$\begin{aligned}
&- x\frac{\partial}{\partial x} F(x,\xi) = -x\frac{\mathrm{d}}{\mathrm{d}x} D_{ln} f(x) + x\frac{\partial}{\partial x} D_{ln} g(\xi x) \\
&= -2D_{ln} f(x) + \big(1-\xi^2\big)x|x| = -2\psi(x) + \big(1-\xi^2\big)x|x|, && (48_{1,5'}) \\
&= \big(1-\xi^2\big)x|x|, && (48_{2,4,2',4'}) \\
&= 2D_{ln} f(x) + \big(1-\xi^2\big)x|x| = 2\big(\varphi(x)-1\big) + \big(1-\xi^2\big)x|x|, && (48_3) \\
&= \frac{1}{2}\big(D_{ln} f(x)\big)^2 + \big(1-\xi^2/2\big)x|x| = \frac{1}{2}\big(\varphi(x)-1\big)^2 + \big(1-\xi^2/2\big)x|x|, && (48_5) \\
&= -2D_{ln} f(x) + 2 + \big(1-\xi^2\big)x|x| = -2\varphi(x) + \big(1-\xi^2\big)x|x|, && (48_{1'}) \\
&= 2D_{ln} f(x) - 2 + \big(1-\xi^2\big)x|x| = 2\big(\psi(x)-1\big) + \big(1-\xi^2\big)x|x|; && (48_{3'})
\end{aligned}$$

and

$$\xi'_{cr} = -\frac{\frac{\partial}{\partial x} F(x,\xi)}{\frac{\partial}{\partial \xi} F(x,\xi)} = \frac{-x\frac{\partial}{\partial x} F(x,\xi)}{x\frac{\partial}{\partial \xi} F(x,\xi)}. \tag{49}$$

The denominator of the last fraction is positive. So, the sign of $\xi'_{cr}(x)$ is the same as that of expression (48).

5 Properties of the negative critical points

Theorem 1. *There are no critical points for negative x in problems* P1, P2, P4, P5, P1′, P2′, P4′, P5′.

Proof. Case 4 *and Case* P2′. By the Mean Value Theorem we derive

$$F(x;\xi) = \varphi(x) - \varphi(\xi x) = \varphi'(\vartheta_1)(1-\xi)x,\ \vartheta_1 < 0.$$

Since $\varphi' < 0$ (see, Proposition 3), we get $F(x;\xi) > 0$ for $x < 0$.

Case 2 *and Case* P4′. If $\xi = 0$, then $F(x;0) = \psi(x) > 0$ for $x < 0$, otherwise $F(x;\xi) = \psi(x) - \psi(\xi x) = \psi'(\vartheta_2)(1-\xi)x > 0,\ \vartheta_2 < 0$.

Case 5. In this case, $F(x;\xi) = \varphi(x) - 2\varphi(\xi x/2) + 1 = \varphi(x) - \varphi(\xi x/2) + \varphi(0) - \varphi(\xi x/2) = \varphi'(\vartheta_3)(1-\xi/2)x + \varphi'(\vartheta_4)x > 0,\ \vartheta_3 < 0, \vartheta_4 < 0$.

Case P1′. From Corollary 10 it follows that $F(x;\xi) = \varphi(x) - \psi(\xi x) + 1 > \psi(x) - \psi(\xi x) + 1 = \psi'(\vartheta_5)(1-\xi)x + 1 \geqslant 1 > 0,\ \vartheta_5 \leqslant 0$.

Case 1 *and Case* P5′. If $\xi = 1$, then $F(x;1) = \psi(x) - \varphi(x) + 1 > 0$ for $x < 0$ (see, Corollary 10), or else $F(x;\xi) = \psi(x) + 1 - \varphi(\xi x) > \varphi(x) - \varphi(\xi x) = \varphi'(\vartheta_6)(1-\xi)x > 0,\ \vartheta_6 < 0$. ☐

Corollary 13. *For problems* P1, P2, P4, P5, P1′, P2′, P4′, P5′*, the characteristic function is a decreasing function with negative x (see,* (41)*).*

Theorem 2. *For problem* P3*, there exists a negative critical point if and only if $\xi \in (\sqrt{3}/3, 1)$ and this unique negative critical point x_- is the minimum point, and $\gamma(x_-;\xi) > 0$.*

Proof. The function $F(x;\xi) = G(x;\xi) - 1$, $\xi \in [0,1]$. The proof follows from the properties of function G. The minimum point $x_- = -x_{min}$ (see, Lemma 1). ☐

Corollary 14. *Let us consider the characteristic function γ for negative x of problem* P3*. Then γ is*

1) *a decreasing function for $\xi \in [0, \sqrt{3}/3]$;*

2) *a decreasing function for $x \in (-\infty, x_-)$ and an increasing function for $x \in (x_-, 0)$ as $\xi \in (\sqrt{3}/3, 1)$;*

3) *an increasing function for $\xi = 1$.*

We have $\lim_{x\to 0} \gamma(x;\xi) = 1$ for all ξ, and if $\xi < 1$, then $\lim_{x\to-\infty} \gamma(x;\xi) = +\infty$, or else $\lim_{x\to-\infty} \gamma(x;1) = 0$.

Proposition 10. *In Case* 3*, the function ξ_{cr} is a decreasing function and*

$$\lim_{x\to-\infty} \xi_{cr}(x) = 1, \quad \lim_{x\to 0} \xi_{cr}(x) = \sqrt{3}/3.$$

Proof.

$$-x\frac{\partial}{\partial x}F'(x;\xi) = -xG'(x;\xi) = \xi x G_1(-\xi x)\bigl(H(x;\xi)-1\bigr) < 0.$$

□

Theorem 3. *For problem* P3′*, there exists one negative critical point for all* $\xi \in (0,1)$ *and this unique negative critical point* $\tilde{x}_-$ *is the minimum point and* $\gamma(\tilde{x}_-;\xi) > 0$*. For* $\xi = 1$ *there are no negative critical points.*

Proof. The function $F(x;\xi) = -\tilde{G}(x;\xi) - 1$, $\xi \in (0,1]$. The proof follows from the properties of function $\tilde{\tilde{G}}$. The minimum point $\tilde{x}_- = -\tilde{x}_{min}$ (see, Lemma 2). □

Corollary 15. *Let us consider the characteristic function* γ *for negative* x *in Problem* P3′*. If* $\xi \in (0,1)$*, then* γ *is a decreasing function for* $x \in (-\infty, \tilde{x}_-)$ *and an increasing function for* $x \in (\tilde{x}_-, 0)$*. The function* $\gamma(x;1)$ *is an increasing function for* $x < 0$*. We obtain* $\lim\limits_{x\to -0}\gamma(x;\xi) = +\infty$ *for all* $\xi \in (0,1]$*,* $\lim\limits_{x\to-\infty}\gamma(x;\xi) = +\infty$ *for* $\xi \in (0,1)$ *and* $\lim\limits_{x\to-\infty}\gamma(x;1) = 0$*.*

Proposition 11. *In Case* $3'$*, the function* ξ_{cr} *is a decreasing function for* $x < 0$*,* $\lim\limits_{x\to-\infty}\xi_{cr}(x) = 1$ *and the graph of this function intersects (in the limit)* x*-axis at the point* $x = -x_*$ *where* x_* *is the positive root of the equation* $x\tanh x = 2$ *or equation* $\tilde{G}(x;0) = -1$*. There are no critical points for* $x \in [x_*, 0)$*.*

Proof.

$$-x\frac{\partial}{\partial x}F'(x;\xi) = x\tilde{G}'(x;\xi) = -xG_1(-x)\bigl(\tilde{H}(x;\xi)-1\bigr) < 0.$$

□

Remark 10. *We will investigate nonnegative critical points in the next article.*

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